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**Gravitational model of the three elements theory: mathematical details calculations**

This document is intended to collect each detailed calculations regarding the mathematical basis of the gravitational model of the three elements theory.

# 1. GEODESIC TIME-LINE EQUATIONS FOR THE MINKOWSKIAN AND RIEMANNIAN METRICS

The aim of this calculation is to prove that time line geodesics in the Minkowskian metric are time line geodesics in the Riemannian metric.

The method is the following.

1) The geodesic equations of the Minkowskian metric are written.
2) They are simplified using those restrictions.
   a) A specific system of inertial reference frames (bases) is used. In this system of bases, the metric matrixes are diagonal. It is the classical “normal map” of General Relativity.
   b) It is supposed that only weak space-time deformations occurs.
   c) It is supposed that infinitely far away from the location of our study, the deformations are null (like in the case of the Schwarzchild metric study).
   d) Only time line trajectories are studied. (What is called “time line” here is a coordinate curve in the “exponential map” of the metric, that is, a time coordinate curve in the system of bases in which the metric matrixes are diagonal). Let’s remind that these are the trajectories of a free falling particle initially at rest when located infinitely far, (and with a null mass at rest).
3) Steps 1) and 2) are executed with the Riemannian metric. This metric is constructed by inversion of the Minkowskian coefficients in its diagonal matrixes and of course opposite sign for the space coefficients. Caution must be applied : 1) the system of bases in which those Riemannian metric matrixes are diagonal might not be the same as the corresponding one in the Minkowskian metric. 2) Rigorously speaking, it must be proven that such a Riemannian metric can be constructed and coherent in such a way. Also, let’s remind that a time line in the Riemannian metric is still a geodesic in this metric but might not be the trajectory af a free falling particle.
4) For allowing further comparison, Minkowskian metric geodesic equations (found at the end of step 2) are written using the Riemaniann metric coefficients and the Riemannian local time.
5) Comparison of Minkowskian time line equations (yielded at the end of step 4) and Riemannian time line (yielded at the end of step 3) equations is done. The aim result is that they must be the same in the approximation used context.

Therefore, more precisely, the aim is proving that, in the Riemannian metric, a time line is approximately also a time line in the Minkowskian metric, and vice versa, in the specific case of the weak space-time deformations.
The restrictions are the weak space-time deformations case. This restriction has no incidence since until now only time line trajectories in the weak deformations case has been used in the study of the gravitational model of the three elements theory.

## 1) Riemannian metric

$$\boxed{\frac{\partial^2 x^\lambda}{\partial \tau^2} = -\Gamma^\lambda_{\mu\nu} \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \tau}} \qquad \textbf{(1)}$$

$$\begin{aligned}
\partial^2 x^0 &= -\Gamma^0_{\mu\nu} dx^\mu dx^\nu \\
&= -\Gamma^0_{00} dx^0 dx^0 - \Gamma^0_{01} dx^0 dx^1 - \Gamma^0_{10} dx^1 dx^0 + \Gamma^0_{11} dx^1 dx^1 \\
&= -\Gamma^0_{00} dx^{0\,2} - \left(\Gamma^0_{01} + \Gamma^0_{10}\right) dx^0 dx^1 - \Gamma^0_{11} dx^{1\,2}
\end{aligned}$$

$$\begin{aligned}
\partial^2 x^1 &= -\Gamma^1_{\mu\nu} dx^\mu dx^\nu \\
&= -\Gamma^1_{00} dx^0 dx^0 - \Gamma^1_{01} dx^0 dx^1 - \Gamma^1_{10} dx^1 dx^0 + \Gamma^1_{11} dx^1 dx^1 \\
&= -\Gamma^1_{00} dx^{0\,2} - \left(\Gamma^1_{01} + \Gamma^1_{10}\right) dx^0 dx^1 - \Gamma^1_{11} dx^{1\,2}
\end{aligned}$$

$$\frac{\partial^2 x^0}{\partial \tau^2} = -\Gamma^0_{00} \left(\frac{\partial x^0}{\partial \tau}\right)^2 - \left(\Gamma^0_{01} + \Gamma^0_{10}\right) \frac{\partial x^0}{\partial \tau} \frac{\partial x^1}{\partial \tau} - \Gamma^0_{11} \left(\frac{\partial x^1}{\partial \tau}\right)^2$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\Gamma^1_{00} \left(\frac{\partial x^0}{\partial \tau}\right)^2 - \left(\Gamma^1_{01} + \Gamma^1_{10}\right) \frac{\partial x^0}{\partial \tau} \frac{\partial x^1}{\partial \tau} - \Gamma^1_{11} \left(\frac{\partial x^1}{\partial \tau}\right)^2$$

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left( \frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right)$$

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{2} g^{00} \left( \frac{\partial g_{00}}{\partial x^0} + \frac{\partial g_{00}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^0} \right) \\
&\quad + \frac{1}{2} g^{01} \left( \frac{\partial g_{10}}{\partial x^0} + \frac{\partial g_{10}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^1} \right) \\
&= \frac{1}{2} g^{00} \left( \frac{\partial g_{00}}{\partial x^0} + \frac{\partial g_{00}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^0} \right) \\
&= \frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^0}
\end{aligned}$$

$$\Gamma^1_{00} = \frac{1}{2} g^{10} \left( \frac{\partial g_{00}}{\partial x^0} + \frac{\partial g_{00}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^0} \right)$$

$$+ \frac{1}{2} g^{11} \left( \frac{\partial g_{10}}{\partial x^0} + \frac{\partial g_{10}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^1} \right)$$

$$= \frac{1}{2} g^{11} \left( 2 \frac{\partial g_{10}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^1} \right)$$

$$\Gamma^0_{01} = \frac{1}{2} g^{00} \left( \frac{\partial g_{00}}{\partial x^1} + \frac{\partial g_{01}}{\partial x^0} - \frac{\partial g_{01}}{\partial x^0} \right)$$

$$+ \frac{1}{2} g^{01} \left( \frac{\partial g_{10}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^0} - \frac{\partial g_{01}}{\partial x^1} \right)$$

$$= \frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^1}$$

$$\Gamma^0_{10} = \frac{1}{2} g^{00} \left( \frac{\partial g_{01}}{\partial x^0} + \frac{\partial g_{00}}{\partial x^1} - \frac{\partial g_{10}}{\partial x^0} \right)$$

$$+ \frac{1}{2} g^{01} \left( \frac{\partial g_{01}}{\partial x^0} + \frac{\partial g_{00}}{\partial x^1} - \frac{\partial g_{10}}{\partial x^1} \right)$$

$$= \frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^1}$$

$$\Gamma^1_{01} = \frac{1}{2} g^{10} \left( \frac{\partial g_{00}}{\partial x^1} + \frac{\partial g_{01}}{\partial x^0} - \frac{\partial g_{01}}{\partial x^0} \right)$$

$$+ \frac{1}{2} g^{11} \left( \frac{\partial g_{10}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^0} - \frac{\partial g_{01}}{\partial x^1} \right)$$

$$= \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial x^0}$$

$$\Gamma^1_{10} = \frac{1}{2} g^{00} \left( \frac{\partial g_{01}}{\partial x^0} + \frac{\partial g_{00}}{\partial x^1} - \frac{\partial g_{10}}{\partial x^0} \right)$$

$$+ \frac{1}{2} g^{11} \left( \frac{\partial g_{11}}{\partial x^0} + \frac{\partial g_{10}}{\partial x^1} - \frac{\partial g_{10}}{\partial x^1} \right)$$

$$= \frac{1}{2} g^{00} \left( \frac{\partial g_{00}}{\partial x^1} \right) + \frac{1}{2} g^{11} \left( \frac{\partial g_{11}}{\partial x^0} \right)$$

$$\Gamma^0_{11} = \frac{1}{2} g^{00} \left( \frac{\partial g_{01}}{\partial x^1} + \frac{\partial g_{01}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^0} \right)$$

$$+ \frac{1}{2} g^{01} \left( \frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1} \right)$$

$$= \frac{1}{2} g^{00} \left( 2 \frac{\partial g_{01}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^0} \right)$$

$$\Gamma^1_{11} = \frac{1}{2} g^{10} \left( \frac{\partial g_{01}}{\partial x^1} + \frac{\partial g_{01}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^0} \right) + \frac{1}{2} g^{11} \left( \frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1} \right) = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial x^1}$$

$$\frac{\partial^2 x^0}{\partial \tau^2} = -\frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^0} \left( \frac{\partial x^0}{\partial \tau} \right)^2 - g^{00} \frac{\partial g_{00}}{\partial x^1} \frac{\partial x^0}{\partial \tau} \frac{\partial x^1}{\partial \tau} - \frac{1}{2} g^{00} \left( 2 \frac{\partial g_{01}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^0} \right) \left( \frac{\partial x^1}{\partial \tau} \right)^2$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{1}{2} g^{11} \left( 2 \frac{\partial g_{10}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^1} \right) \left( \frac{\partial x^0}{\partial \tau} \right)^2 - g^{11} \frac{\partial g_{11}}{\partial x^0} \frac{\partial x^0}{\partial \tau} \frac{\partial x^1}{\partial \tau} - \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial x^1} \left( \frac{\partial x^1}{\partial \tau} \right)^2$$

Extradiagonal metric coefficients are null *everywhere*, because we can choose such a system of bases which yield diagonal metric matrixes. Therefore their derivatives are null and we get the following.

$$\frac{\partial^2 x^0}{\partial \tau^2} = -\frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^0} \left( \frac{\partial x^0}{\partial \tau} \right)^2 - g^{00} \frac{\partial g_{00}}{\partial x^1} \frac{\partial x^0}{\partial \tau} \frac{\partial x^1}{\partial \tau} + \frac{1}{2} g^{00} \frac{\partial g_{11}}{\partial x^0} \left( \frac{\partial x^1}{\partial \tau} \right)^2$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = \frac{1}{2} g^{11} \frac{\partial g_{00}}{\partial x^1} \left( \frac{\partial x^0}{\partial \tau} \right)^2 - g^{11} \frac{\partial g_{11}}{\partial x^0} \frac{\partial x^0}{\partial \tau} \frac{\partial x^1}{\partial \tau} - \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial x^1} \left( \frac{\partial x^1}{\partial \tau} \right)^2$$

$$\frac{\partial^2 x^0}{\partial \tau^2} = -\frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^0} \left( \frac{\partial x^0}{\partial \tau} \right)^2 - g^{00} \frac{\partial g_{00}}{\partial x^1} \frac{\partial x^0}{\partial \tau} \frac{\partial r}{\partial \tau} - \frac{1}{2} g^{00} \left( 2 \frac{\partial g_{01}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^0} \right) \left( \frac{\partial r}{\partial \tau} \right)^2 = -\frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^0} \left( \frac{\partial x^0}{\partial \tau} \right)^2$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{1}{2} g^{11} \left( 2 \frac{\partial g_{10}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^1} \right) \left( \frac{\partial x^0}{\partial \tau} \right)^2 - g^{11} \frac{\partial g_{11}}{\partial x^0} \frac{\partial x^0}{\partial \tau} \frac{\partial r}{\partial \tau} - \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial x^1} \left( \frac{\partial r}{\partial \tau} \right)^2 = \frac{1}{2} g^{11} \left( \frac{\partial g_{00}}{\partial x^1} - 2 \frac{\partial g_{10}}{\partial x^0} \right) \left( \frac{\partial x^0}{\partial \tau} \right)^2 = \frac{1}{2} g^{11} \frac{\partial g_{00}}{\partial x^1} \left( \frac{\partial x^0}{\partial \tau} \right)^2$$

It has been used $g_{10} = 0$. Of course, it is always possible to find a system of bases in which the metric matrixes are diagonal. Therefore, the derivative of those extra-diagonal coefficients is always null.

It has been used also $\partial x^1 / \partial \tau = \partial r / \partial \tau \simeq 0$ along the considered trajectory.

This is because only a particular case of trajectories will be studied. This case is a free falling particle initially with a null speed and located infinitively far from the attracting mass, (with a null mass).

Therefore, the equations (1) above becomes the following.

$$\frac{\partial^2 x^0}{\partial \tau^2} = -\frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^0} \left( \frac{\partial x^0}{\partial \tau} \right)^2 \qquad \textbf{(2)}$$
$$\frac{\partial^2 x^1}{\partial \tau^2} = \frac{1}{2} g^{11} \frac{\partial g_{00}}{\partial x^1} \left( \frac{\partial x^0}{\partial \tau} \right)^2$$

And this is the maximum trajectories in the pseudo-riemannian Minkowskian metric. They are known to be free falling particle's trajectories.

Now replacing *g* by *h* and $\tau$ by $\tau$' we get the equations of the minimal trajectories in the Riemannian metric:

$$\frac{\partial^2 x^0}{\partial \tau'^2} = -\frac{1}{2} h^{00} \frac{\partial h_{00}}{\partial x^0} \left( \frac{\partial x^0}{\partial \tau'} \right)^2 \qquad \textbf{(3)}$$
$$\frac{\partial^2 x^1}{\partial \tau'^2} = \frac{1}{2} h^{11} \frac{\partial h_{00}}{\partial x^1} \left( \frac{\partial x^0}{\partial \tau'} \right)^2$$

Those equations are no longer the free falling particle's trajectories. But the aim of this note is to prove that they remain a good approximation of the free falling particle's trajectories in the case of the weak space-time deformations.

For this to be proven, we start from equations (2), and we substitute the *g* coefficients by the *h* coefficients, and the $\tau$ variable by the $\tau$' variable.

## 2) Minkowskian metric

Therefore, let's start from equations (2) :

$$\frac{\partial^2 x^0}{\partial \tau^2} = -\frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^0} \left( \frac{\partial x^0}{\partial \tau} \right)^2$$
$$\frac{\partial^2 x^1}{\partial \tau^2} = \frac{1}{2} g^{11} \frac{\partial g_{00}}{\partial x^1} \left( \frac{\partial x^0}{\partial \tau} \right)^2$$

Now let's express those equations with the help of the Riemannian *h* coefficients. We get the following relationships between the two metric *g* and *h* coefficients.

$$g^{00} = 1/g_{00} = h_{00} \qquad g^{11} = 1/g_{11} = -h_{11} \qquad g_{00} = 1/g^{00} = h^{00} = 1/h_{00}$$

Therefore, the Minkowskian extremal trajectories now follow those two equation rules.

$$\frac{\partial^2 x^0}{\partial \tau^2} = -\frac{1}{2} h_{00} \frac{\partial\left(1/h_{00}\right)}{\partial x^0}\left(\frac{\partial x^0}{\partial \tau}\right)^2$$

$$= \frac{1}{2} h_{00} / h_{00}{}^2 \frac{\partial h_{00}}{\partial x^0}\left(\frac{\partial x^0}{\partial \tau}\right)^2$$

$$= \frac{1}{2} 1 / h_{00} \frac{\partial h_{00}}{\partial x^0}\left(\frac{\partial x^0}{\partial \tau}\right)^2$$

$$= \frac{1}{2} h^{00} \frac{\partial h_{00}}{\partial x^0}\left(\frac{\partial x^0}{\partial \tau}\right)^2$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = \frac{1}{2} g^{11} \frac{\partial g_{00}}{\partial x^1}\left(\frac{\partial x^0}{\partial \tau}\right)^2$$

$$= -\frac{1}{2} h_{11} \frac{\partial\left(1/h_{00}\right)}{\partial x^1}\left(\frac{\partial x^0}{\partial \tau}\right)^2$$

$$= \frac{1}{2} h_{11} / h_{00}{}^2 \frac{\partial h_{00}}{\partial x^1}\left(\frac{\partial x^0}{\partial \tau}\right)^2$$

$$= \frac{1}{2} h_{11}^3 \frac{\partial h_{00}}{\partial x^1}\left(\frac{\partial x^0}{\partial \tau}\right)^2$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = \frac{1}{2} h_{11}^3 \frac{\partial h_{00}}{\partial x^1}\left(\frac{\partial x^0}{\partial \tau}\right)^2$$

Finally, we get the following.

$$\frac{\partial^2 x^0}{\partial \tau^2} = \frac{1}{2} h^{00} \frac{\partial h_{00}}{\partial x^0}\left(\frac{\partial x^0}{\partial \tau}\right)^2 \qquad \textbf{(5)}$$

$$\frac{\partial^2 x^1}{\partial \tau^2} \simeq \frac{1}{2} h_{11}^3 \frac{\partial h_{00}}{\partial x^1}\left(\frac{\partial x^0}{\partial \tau}\right)^2 \qquad \textbf{(6)}$$

Now let's replace this Minkowskian local time $\tau$ by the Riemannian local time $\tau'$. We get the following relationships.

$$ds = \sqrt{g_{00} dx^{0\,2} + g_{11} dx^{1\,2}} \qquad ds' = \sqrt{h_{00} dx^{0\,2} + h_{11} dx^{1\,2}}$$

$$= \sqrt{g_{00} c^2 dt^2 + g_{11} dr^2} \qquad = \sqrt{h_{00} c^2 dt^2 + h_{11} dr^2}$$

$$d\tau = \sqrt{g_{00}}\, dt \qquad d\tau' = \sqrt{h_{00}}\, dt$$

$$\frac{\partial \tau'}{\partial \tau} = \frac{\sqrt{h_{00}}\, dt}{\sqrt{g_{00}}\, dt} = \sqrt{\frac{h_{00}}{g_{00}}} = \sqrt{h_{00}{}^2} = h_{00}$$

$$\frac{\partial x^0}{\partial \tau} = \frac{\partial \tau'}{\partial \tau}\frac{\partial x^0}{\partial \tau'} = h_{00}\frac{\partial x^0}{\partial \tau'} \qquad \textbf{(7)}$$

$$\frac{\partial^2 x^0}{\partial \tau^2} = \frac{\partial}{\partial \tau}\left(\frac{\partial x^0}{\partial \tau}\right)$$

$$= \frac{\partial \tau'}{\partial \tau}\frac{\partial}{\partial \tau'}\left(\frac{\partial \tau'}{\partial \tau}\frac{\partial x^0}{\partial \tau'}\right)$$

$$= h_{00}\frac{\partial}{\partial \tau'}\left(h_{00}\frac{\partial x^0}{\partial \tau'}\right)$$

$$= h_{00}\frac{\partial h_{00}}{\partial \tau'}\frac{\partial x^0}{\partial \tau'} + h_{00}{}^2\frac{\partial^2 x^0}{\partial \tau'^2}$$

$$\frac{\partial^2 x^0}{\partial \tau^2} = h_{00}\frac{\partial h_{00}}{\partial \tau'}\frac{\partial x^0}{\partial \tau'} + h_{00}{}^2\frac{\partial^2 x^0}{\partial \tau'^2} \qquad \textbf{(8)}$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = h_{00}\frac{\partial h_{00}}{\partial \tau'}\frac{\partial x^1}{\partial \tau'} + h_{00}{}^2\frac{\partial^2 x^1}{\partial \tau'^2} \qquad \textbf{(9)}$$

Therefore, using equations (5), (6), (8), and (9), the extremal trajectories **first equation** (5) becomes now the following.

$$h_{00}\frac{\partial h_{00}}{\partial \tau'}\frac{\partial x^0}{\partial \tau'} + h_{00}{}^2\frac{\partial^2 x^0}{\partial \tau'^2} = \frac{1}{2}h^{00}\frac{\partial h_{00}}{\partial x^0}\left(h_{00}\frac{\partial x^0}{\partial \tau'}\right)^2$$

$$h_{00}\frac{\partial h_{00}}{\partial \tau'}\frac{\partial x^0}{\partial \tau'} + h_{00}{}^2\frac{\partial^2 x^0}{\partial \tau'^2} = \frac{1}{2}h_{00}\frac{\partial h_{00}}{\partial x^0}\left(\frac{\partial x^0}{\partial \tau'}\right)^2$$

$$\frac{\partial^2 x^0}{\partial \tau'^2} = \frac{1}{2}h^{00}\frac{\partial h_{00}}{\partial x^0}\left(\frac{\partial x^0}{\partial \tau'}\right)^2 - h^{00}\frac{\partial h_{00}}{\partial \tau'}\frac{\partial x^0}{\partial \tau'}$$

$$= \frac{1}{2}h^{00}\frac{\partial h_{00}}{\partial x^0}\left(\frac{\partial x^0}{\partial \tau'}\right)^2 - h^{00}\frac{\partial h_{00}}{\partial x^0}\frac{\partial x^0}{\partial \tau'}\frac{\partial x^0}{\partial \tau'}$$

$$= \frac{1}{2}h^{00}\frac{\partial h_{00}}{\partial x^0}\left(\frac{\partial x^0}{\partial \tau'}\right)^2 - h^{00}\frac{\partial h_{00}}{\partial x^0}\left(\frac{\partial x^0}{\partial \tau'}\right)^2$$

$$\frac{\partial^2 x^0}{\partial \tau'^2} = -\frac{1}{2}h^{00}\frac{\partial h_{00}}{\partial x^0}\left(\frac{\partial x^0}{\partial \tau'}\right)^2 \qquad \textbf{(10)}$$

Let’s notice the last simplification and the same final result as the first equation of the Riemannian metric geodesics.
That is to say that the first equation of (2) is equivalent to the first equation of (3).

The extremal trajectories **second equation** (6) becomes now the following.

$$h_{00}\frac{\partial h_{00}}{\partial \tau'}\frac{\partial x^1}{\partial \tau'} + h_{00}{}^2\frac{\partial^2 x^1}{\partial \tau'^2} = \frac{1}{2}h_{11}^3\frac{\partial h_{00}}{\partial x^1}\left(\frac{\partial x^0}{\partial \tau}\right)^2$$

$$h_{00}\frac{\partial h_{00}}{\partial \tau'}\frac{\partial x^1}{\partial \tau'} + h_{00}{}^2\frac{\partial^2 x^1}{\partial \tau'^2} = \frac{1}{2}h_{11}^3\frac{\partial h_{00}}{\partial x^1}\left(h_{00}\frac{\partial x^0}{\partial \tau'}\right)^2$$

$$\frac{\partial^2 x^1}{\partial \tau'^2} = \frac{1}{2} h_{11}^3 \frac{\partial h_{00}}{\partial x^1} \left( \frac{\partial x^0}{\partial \tau'} \right)^2 - h_{11} \frac{\partial h_{00}}{\partial \tau'} \frac{\partial x^1}{\partial \tau'}$$

$$\frac{\partial^2 x^1}{\partial \tau'^2} = \frac{1}{2} h_{11}^3 \frac{\partial h_{00}}{\partial x^1} \left( \frac{\partial x^0}{\partial \tau'} \right)^2 \qquad \textbf{(11)}$$

Finally, with (10) and (11), we get the following equations of the Minkowskian metric extremal trajectories expressed with the help of the *h* Riemannian coefficients and the τ' Riemannian local time :

$$\boxed{\begin{aligned} \frac{\partial^2 x^0}{\partial \tau'^2} &= -\frac{1}{2} h^{00} \frac{\partial h_{00}}{\partial x^0} \left( \frac{\partial x^0}{\partial \tau'} \right)^2 \\ \frac{\partial^2 x^1}{\partial \tau'^2} &= \frac{1}{2} h_{11}^3 \frac{\partial h_{00}}{\partial x^1} \left( \frac{\partial x^0}{\partial \tau'} \right)^2 \end{aligned}} \qquad \textbf{(12)}$$

**Minkowskian metric extremal trajectories expressed with Riemannian *h* coefficients and Riemannian τ' local time**

## 3) Comparison

Now it remains to compare those equations (12), with the equations of the minimal trajectories in the Riemannian metric, namely equations (3) :

$$\boxed{\begin{aligned} \frac{\partial^2 x^0}{\partial \tau'^2} &= -\frac{1}{2} h^{00} \frac{\partial h_{00}}{\partial x^0} \left( \frac{\partial x^0}{\partial \tau'} \right)^2 \\ \frac{\partial^2 x^1}{\partial \tau'^2} &\simeq \frac{1}{2} h_{11}^{-1} \frac{\partial h_{00}}{\partial x^1} \left( \frac{\partial x^0}{\partial \tau'} \right)^2 \end{aligned}} \qquad \textbf{(13)}$$

**Riemannian metric extremal trajectories**

The final work consist only of comparing equations (12), and equations (13).
The first equation of (12) is exactly the same as the first equation of (13).
Therefore, the only work to do is comparing the last equation of (12), and the last equation of (13).
Those last equations are the same if $h_{11} \simeq 1$, which is always true in the studied cases.

**As a conclusion, the time line geodesics in the Minkowskian metric are approximately the time line geodesics in the Riemannian metric. The used case for these approximations is the weak space-time deformations case.**

## 2. EXISTENCE OF THE SEARCHED RIEMANNIAN METRIC

When constructing the Riemannian metric in this note, the correctness of this construction must be proven.
For example, if we suppose that this Riemannian metric is such as its matrixes are diagonal in the same system of diagonalising bases as the Minkowskian one, then this metric is not legal. Indeed, the calculations above has proven that, if this could be a valid Riemanninan metric, then its time lines would not be exactly geodesics (but only approximatively). But this approximation is not enough from a mathematical point of view, because it should be exactly geodesics. If this is not the case, then it means that the constructed metric is not a valid one. In other words, it means that, when a Riemannian metric is searched such as its coefficients are inversed with respect to the Minkowskian diagonalised one (and with opposed sign for the space coefficients), then it generates non nul extra-diagonal coefficients. That's for sure. However, it is still unclear even with some new extra-diagonal non null coefficients, that the resulting Riemannian metric is still a valid one.
For proving the validity of this Riemannian metric, it will be supposed that the existence of this metric is equivalent to the existence of solutions for its geodesic equations (that is, equivalent to the validity of the corresponding geodesics).
Therefore, the calculations will be quite straight forward. We just write the geodesics equations resulting from this supposed Riemannian metric, and we try to prove the existence of solution for each initial case.

Those equations are the following.

$$\frac{\partial^2 x^\lambda}{\partial \tau^2} = -\Gamma^\lambda_{\mu\nu} \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \tau}$$

It is not possible to simplify immediately this system, because extra-diagonal coefficients are no longer null everywhere. But diagonal coefficients are known, and only the extra-diagonal coefficients are unknown.
Therefore, this system is a $n$ equations system implying the first derivatives of $n(n-1)/2$ unknown variables.
An immediate remark can be done. The ratio of this number of unknown variables divided by the number of equations is equal to $n(n-1)/(2n) = (n-1)/2$, therefore it increase with the number of involved dimensions, $n$. In other words, the studied case of this note, which is the case of a spherical symmetry in space and which implies only $n = 2$, is probably one of the worst case. Therefore, only this case will be studied. That's saying there are only 2 equations, and only dimension 0 and 1 :

$$\frac{\partial^2 x^0}{\partial \tau^2} = -\Gamma^0_{\mu\nu} \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \tau}$$
$$\frac{\partial^2 x^1}{\partial \tau^2} = -\Gamma^1_{\mu\nu} \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \tau}$$

$$\Gamma^0_{\mu\upsilon} = \frac{1}{2} h^{0m} \left( \frac{\partial h_{m\mu}}{\partial x^\upsilon} + \frac{\partial h_{m\upsilon}}{\partial x^\mu} - \frac{\partial h_{\mu\upsilon}}{\partial x^m} \right)$$
$$\Gamma^1_{\mu\upsilon} = \frac{1}{2} h^{1m} \left( \frac{\partial h_{m\mu}}{\partial x^\upsilon} + \frac{\partial h_{m\upsilon}}{\partial x^\mu} - \frac{\partial h_{\mu\upsilon}}{\partial x^m} \right)$$

Now let's take the case of the Schwarzchild metric. Let's write *r* the distance from the attracting mass, as usual.
Each times a $h^{\alpha\beta}$ or a $h_{\alpha\beta}$ coefficient is a known function of *r* when $\alpha=\beta$, and becomes a unknown variable when $\alpha\neq\beta$. Here, with $n=2$, there is only one unknown variable which is $h_{01}=h_{10}$, and we have :
$h_{00}=\left(1-\frac{M}{r}\right)^{-1}$, and $h_{11}=1-\frac{M}{r}$.
Also, one way of simplification is as usual to fix the time parameter $\tau$. Indeed,

Each term like $\frac{\partial g_{\alpha\alpha}}{\partial x^{\gamma}}$ can be written

$$\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}=\frac{\partial g_{\alpha\beta}}{\partial r}\frac{\partial r}{\partial x^{\gamma}}=\frac{\partial g_{\alpha\beta}}{\partial r}\frac{\partial}{\partial x^{\gamma}}\left(\sqrt{g_{00}\left(\frac{\partial x^0}{\partial\tau}\right)^2+2g_{01}\frac{\partial x^0}{\partial\tau}\frac{\partial x^1}{\partial\tau}+g_{11}\left(\frac{\partial x^1}{\partial\tau}\right)^2}\right)$$

For example :

$$\frac{\partial g_{00}}{\partial x^{0}}=\frac{\partial g_{00}}{\partial r}\frac{\partial}{\partial r}\left(\int\sqrt{g_{00}\left(\frac{\partial x^0}{\partial\tau}\right)^2+2g_{01}\frac{\partial x^0}{\partial\tau}\frac{\partial x^1}{\partial\tau}+g_{11}\left(\frac{\partial x^1}{\partial\tau}\right)^2}\,d\tau\right)$$

$$=\frac{1}{2}\frac{\partial g_{00}}{\partial r}\int\left(g_{00}\left(\frac{\partial x^0}{\partial\tau}\right)^2+2g_{01}\frac{\partial x^0}{\partial\tau}\frac{\partial x^1}{\partial\tau}+g_{11}\left(\frac{\partial x^1}{\partial\tau}\right)^2\right)^{-\frac{1}{2}}\frac{\partial}{\partial r}\left(g_{00}\left(\frac{\partial x^0}{\partial\tau}\right)^2+2g_{01}\frac{\partial x^0}{\partial\tau}\frac{\partial x^1}{\partial\tau}+g_{11}\left(\frac{\partial x^1}{\partial\tau}\right)^2\right)d\tau$$

$$=\frac{1}{2}\frac{\partial g_{00}}{\partial r}\int\frac{\partial}{\partial r}\left(g_{00}\left(\frac{\partial x^0}{\partial\tau}\right)^2+2g_{01}\frac{\partial x^0}{\partial\tau}\frac{\partial x^1}{\partial\tau}+g_{11}\left(\frac{\partial x^1}{\partial\tau}\right)^2\right)d\tau$$

Therefore, $\frac{\partial g_{00}}{\partial x^0}$ has terms of the form $\frac{\partial g_{01}}{\partial x^0}$ which is unknown.
As a conclusion, the differential system of equations is quite complicated.
The problem is much easier to address from the opposite sense. This is the method adopted in this note. The Minkowskian coefficients are first calculated only in a system of bases in which the matrixes are diagonals. Then the Riemannian diagonal coefficients are constructed from the Minkowskian one, but it is still supposed that those Riemannian matrixes are diagonals. Therefore, the calculations are much easier and the result is that the Riemannian time line geodesics are approximately the Minkowskian ones. But the issue with this method is of course that the two metrics do not share any more the same system of bases which allows those diagonal matrixes.

# 3. EXTREMUM TYPES

The aim is to prove the sign of each extremum. We must prove that a Minkowskian metric geodesic is a ***maximum*** extremum, and that a Riemannian metric geodesic is a ***minimum*** extremum.

This has been found to be quite heavy to calculate in the general case.
Therefore, this will be proven in the specific case of time line trajectories. Indeed, this is enough for the scope of our study, since it is proven in this note that, a Riemannian time line is approximately a Minkowskian time line and vice versa, in the case of the weak space-time deformations.

Let's write the general equation of an action for some given metric g :

$$A = \int \sqrt{g_{00}\left(\frac{\partial x^0}{\partial \tau}\right)^2 + g_{11}\left(\frac{\partial x^1}{\partial \tau}\right)^2}\, d\tau$$

Let's modify the trajectory in the following way :

$$x^1 \rightarrow x^1 + r\ f(\tau)$$

Where $f(\tau)$ is a function of the $\tau$ parameter, and << $r$ >> is a real variable used to modify the trajectory.
With this modelisation, we get :

$$\frac{\partial A}{\partial r} = \int \frac{\partial}{\partial r}\sqrt{g_{00}\left(\frac{\partial x^0}{\partial \tau}\right)^2 + g_{11}\left(\frac{\partial x^1}{\partial \tau}\right)^2}\, d\tau$$

$$\frac{\partial A}{\partial r} = \int \left( g_{00}\left(\frac{\partial x^0}{\partial \tau}\right)^2 + g_{11}\left(\frac{\partial x^1}{\partial \tau}\right)^2 \right)^{-\frac{1}{2}} \frac{\partial}{\partial r}\left( g_{00}\left(\frac{\partial x^0}{\partial \tau}\right)^2 + g_{11}\left(\frac{\partial x^1}{\partial \tau}\right)^2 \right) d\tau$$

Now the classical method will be used. We force $\tau$ to be the space-time distance parameter, such as

$$\sqrt{g_{00}\left(\frac{\partial x^0}{c\partial \tau}\right)^2 + g_{11}\left(\frac{\partial x^1}{c\partial \tau}\right)^2} = 1$$

And that is the local time parameter as usual.

Therefore, the last equation becomes :

$$\frac{\partial A}{\partial r} = \frac{1}{c}\int \frac{\partial}{\partial r}\left( g_{00}\left(\frac{\partial x^0}{\partial \tau}\right)^2 + g_{11}\left(\frac{\partial x^1}{\partial \tau}\right)^2 \right) d\tau$$

$$\frac{\partial A}{\partial r} = \frac{1}{c}\int \left( \frac{\partial g_{\mu\mu}}{\partial r}\left(\frac{\partial x^\mu}{\partial \tau}\right)^2 + 2 g_{\mu\mu}\frac{\partial x^\mu}{\partial \tau}\frac{\partial^2 x^\mu}{\partial r \partial \tau} \right) d\tau \qquad \text{with } \mu = 0,1$$

Let's try to compare the two metric derivatives of actions, by the way :

$$\frac{\partial A}{\partial r} = \frac{1}{c}\int \left( \frac{\partial g_{00}}{\partial r}\left(\frac{\partial x^0}{\partial \tau}\right)^2 + 2 g_{00}\frac{\partial x^0}{\partial \tau}\frac{\partial^2 x^0}{\partial r \partial \tau} + \frac{\partial g_{11}}{\partial r}\left(\frac{\partial x^1}{\partial \tau}\right)^2 + 2 g_{11}\frac{\partial x^1}{\partial \tau}\frac{\partial^2 x^1}{\partial r \partial \tau} \right) d\tau$$

$$\frac{\partial A}{\partial r}=\frac{1}{c}\int\left(\frac{\partial g_{00}}{\partial r}\left(\frac{c}{\sqrt{g_{00}}}\right)^2+2g_{00}\frac{c}{\sqrt{g_{00}}}\frac{\partial}{\partial r}\left(\frac{c}{\sqrt{g_{00}}}\right)+\frac{\partial g_{11}}{\partial r}\left(\frac{\partial x^1}{\partial\tau}\right)^2+2g_{11}\frac{\partial x^1}{\partial\tau}\frac{\partial}{\partial\tau}\left(\frac{\partial x^1}{\partial r}\right)\right)d\tau$$

$$\frac{\partial A}{\partial r}=\frac{1}{c}\int\left(\frac{\partial g_{00}}{\partial r}\left(\frac{c}{\sqrt{g_{00}}}\right)^2-2g_{00}\frac{c^2}{\sqrt{g_{00}}}\frac{1}{2}\frac{1}{g_{00}^{\frac{3}{2}}}\frac{\partial g_{00}}{\partial r}+\frac{\partial g_{11}}{\partial r}\left(\frac{\partial x^1}{\partial\tau}\right)^2+2g_{11}\frac{\partial x^1}{\partial\tau}\frac{\partial}{\partial\tau}\left(\frac{\partial x^1}{\partial r}\right)\right)d\tau$$

$$\frac{\partial A}{\partial r}=\frac{1}{c}\int\left(\frac{\partial g_{00}}{\partial r}\frac{c^2}{g_{00}}-\frac{c^2}{g_{00}}\frac{\partial g_{00}}{\partial r}+\frac{\partial g_{11}}{\partial r}\left(\frac{\partial x^1}{\partial\tau}\right)^2+2g_{11}\frac{\partial x^1}{\partial\tau}\frac{\partial f(\tau)}{\partial\tau}\right)d\tau$$

$$\frac{\partial A_m}{\partial r}=\frac{1}{c}\int\left(\frac{\partial g_{11}}{\partial r}\left(\frac{\partial x^1}{\partial\tau}\right)^2+2g_{11}\frac{\partial x^1}{\partial\tau}\frac{\partial f(\tau)}{\partial\tau}\right)d\tau$$

Or :

$$\frac{\partial A_r}{\partial r}=\frac{1}{c}\int\left(\frac{\partial h_{11}}{\partial r}\left(\frac{\partial x^1}{\partial\tau'}\right)^2+2h_{11}\frac{\partial x^1}{\partial\tau'}\frac{\partial g(\tau')}{\partial\tau'}\right)d\tau'$$

$$\frac{\partial A_m}{\partial r}=\frac{1}{c}\int\left(\frac{\partial g_{11}}{\partial r}\left(\frac{\partial x^1}{\partial\tau}\right)^2+2g_{11}\frac{\partial x^1}{\partial\tau}\frac{\partial f(\tau)}{\partial\tau}\right)d\tau$$

$$\frac{\partial A_m}{\partial r}=\frac{1}{c}\int\left(\frac{\partial}{\partial r}\left(\frac{-1}{h_{11}}\right)\left(\frac{\partial\tau'}{\partial\tau}\frac{\partial x^1}{\partial\tau'}\right)^2-\frac{2}{h_{11}}\frac{\partial x^1}{\partial\tau'}\frac{\partial\tau'}{\partial\tau}\frac{\partial f(\tau)}{\partial\tau'}\frac{\partial\tau'}{\partial\tau}\right)\frac{\partial\tau}{\partial\tau'}d\tau'$$

$$\frac{\partial A_m}{\partial r}=\frac{1}{c}\int\left(\frac{1}{h_{11}^2}\frac{\partial h_{11}}{\partial r}\left(h_{11}\frac{\partial x^1}{\partial\tau'}\right)^2-\frac{2}{h_{11}}\frac{\partial x^1}{\partial\tau'}h_{11}\frac{\partial f(\tau)}{\partial\tau'}h_{11}\right)h_{11}d\tau'$$

$$\frac{\partial A_m}{\partial r}=\frac{1}{c}\int\left(h_{11}\frac{\partial h_{11}}{\partial r}\left(\frac{\partial x^1}{\partial\tau'}\right)^2-2h_{11}^2\frac{\partial x^1}{\partial\tau'}\frac{\partial f(\tau)}{\partial\tau'}\right)d\tau'$$

To be compared with (above) :

$$\frac{\partial A_r}{\partial r}=\frac{1}{c}\int\left(\frac{\partial h_{11}}{\partial r}\left(\frac{\partial x^1}{\partial\tau'}\right)^2+2h_{11}\frac{\partial x^1}{\partial\tau'}\frac{\partial g(\tau')}{\partial\tau'}\right)d\tau'$$

This is not enough but probably too approximated…. It should also probably need an integration by part and the using of the value of the derivative on the geodesic trajectory (equal to 0)…. To be worked. Anyhow the best for comparing the first derivatives of the actions is to work with the geodesic equations as it is done in this note.

Let's restart from the general equation above :

$$\frac{\partial A}{\partial r}=\frac{1}{c}\int\left(\dot{g}_{\mu\mu}v_\mu^2+2g_{\mu\mu}v_\mu\dot{v}_\mu\right)d\tau \qquad \text{where} \quad \dot{g}_{\mu\mu}=\frac{\partial g_{\mu\mu}}{\partial r},\ v_\mu=\frac{\partial x^\mu}{\partial\tau},\ \dot{v}_\mu=\frac{\partial^2 x^\mu}{\partial r\partial\tau}$$

This is not only true for a geodesic equation but in the general case, therefore, now we can derivate once again :

$$\frac{\partial^2 A}{\partial r^2} = \frac{1}{c}\int\left(\ddot{g}_{\mu\mu}v_\mu^2 + 4\dot{g}_{\mu\mu}v_\mu\dot{v}_\mu + 2g_{\mu\mu}\dot{v}_\mu^2 + 2g_{\mu\mu}v_\mu\ddot{v}_\mu\right)d\tau$$ After a little calculation.

Now, using $c^2 d\tau^2 = g_{00}\left(dx^0\right)^2 + g_{11}\left(dx^1\right)^2$, it should be possible to get the result in the general case. But this gives heavy calculations. Let's calculate this in a more particular case, the time line case.
In this case, we get $\frac{\partial x^1}{\partial \tau} = 0$. Under this condition, the last equation becomes the following.

$$\frac{\partial^2 A}{\partial r^2} = \frac{1}{c}\int\left(\ddot{g}_{00}v_0^2 + 4\dot{g}_{00}v_0\dot{v}_0 + 2g_{00}\dot{v}_0^2 + 2g_{00}v_0\ddot{v}_0 + 2g_{11}\dot{v}_1^2\right)d\tau$$

Now we can use simply :

$$v_0 = \frac{1}{\sqrt{g_{00}}} \qquad \dot{v}_0 = -\frac{1}{2}g_{00}^{-\frac{3}{2}}\dot{g}_{00} \qquad \ddot{v}_0 = \frac{1}{4}g_{00}^{-\frac{5}{2}}\left(3\dot{g}_{00}^2 - 2g_{00}\ddot{g}_{00}\right)$$

Substituting this into the time line action equation above, the temporal term vanishes, and it yields the following equation.

$$\frac{\partial^2 A}{\partial r^2} = \frac{2}{c}\int g_{11}\dot{v}_1^2 d\tau \qquad \textbf{(13')}$$

Now, using a Minkowskian metric, the equation is the same as above, $g$ being the Minkowskian metric coefficient.
The second derivatives of the Minkowskian action is :

$$\frac{\partial^2 A_m}{\partial r^2} = \frac{2}{c}\int g_{11}\dot{v}_1^2 d\tau$$

Therefore, this second derivative of $A_m$ is negative (since $g_{11}$ is negative). Therefore the Minkowskian geodesic is a maximal extremum.
And using a Riemannian metric, equation, the second derivatives of the Riemannian action is :

$$\frac{\partial^2 A_r}{\partial r^2} = \frac{2}{c}\int h_{11}\dot{v}_1^2 d\tau'$$

which can be written, using the g coefficient for comparison, and $d\tau' = h_{00}d\tau$ :

$$\frac{\partial^2 A_r}{\partial r^2} = \frac{2}{c}\int \dot{v}_1^2 d\tau$$

Therefore, this second derivative of $A_r$ is positive and the Riemannian geodesic is a minimal extremum.
The final result is the following :

- The Minkowskian geodesic is a maximal extremum.
- The Riemannian geodesic is a minimal extremum.

# 4. REWRITING USING THE SPACE-TIME SLOPE

Let's rewrite equations (12) (Minkowskian time line geodesic equations) using the space-time slope.
It will be used : $h_{11} = g_{00} = cos^2(\alpha)$ $h_{00} = 1/h_{11} = 1/cos^2(\alpha)$

Equations (12) becomes :

$$\frac{\partial}{\partial\tau'}\left(\frac{\partial x^0}{\partial\tau'}\right) = -\frac{1}{2}cos^2(\alpha)\frac{\partial}{\partial x^0}\left(\frac{1}{cos^2(\alpha)}\right)\left(\frac{\partial ct}{\partial\tau'}\right)^2$$
$$\frac{\partial}{\partial\tau'}\left(\frac{\partial x^1}{\partial\tau'}\right) = \frac{1}{2}\left(cos^2(\alpha)\right)^3\frac{\partial}{\partial x^1}\left(\frac{1}{cos^2(\alpha)}\right)\left(\frac{\partial ct}{\partial\tau'}\right)^2$$

$$\frac{\partial}{\partial\tau'}\left(\frac{\partial ct}{\partial\tau'}\right) = -\frac{1}{2}cos^2(\alpha)\frac{\partial}{\partial ct}\left(\frac{1}{cos^2(\alpha)}\right)\left(\frac{\partial ct}{\partial\tau'}\right)^2$$
$$\frac{\partial}{\partial\tau'}\left(\frac{\partial x}{\partial t}\frac{\partial t}{\partial\tau'}\right) = \frac{1}{2}\left(cos^2(\alpha)\right)^3\frac{\partial}{\partial x}\left(\frac{1}{cos^2(\alpha)}\right)\left(\frac{\partial ct}{\partial\tau'}\right)^2$$

$$\frac{\partial}{\partial\tau'}\left(c\,cos(\alpha)\right) = -\frac{1}{2}cos^2(\alpha)\frac{\partial}{\partial ct}\left(\frac{1}{cos^2(\alpha)}\right)\left(c\,cos(\alpha)\right)^2$$
$$\frac{\partial}{\partial\tau'}\left(\frac{\partial x}{\partial t}cos(\alpha)\right) = \frac{1}{2}\left(cos^2(\alpha)\right)^3\frac{\partial}{\partial x}\left(\frac{1}{cos^2(\alpha)}\right)\left(c\,cos(\alpha)\right)^2$$

$$\frac{\partial t}{\partial\tau'}\frac{\partial\alpha}{\partial t}\frac{\partial}{\partial\alpha}\left(cos(\alpha)\right) = -\frac{1}{2}cos^2(\alpha)\frac{\partial\alpha}{\partial t}\frac{\partial}{\partial\alpha}\left(\frac{1}{cos^2(\alpha)}\right)\left(cos(\alpha)\right)^2$$
$$\frac{\partial\left(cos(\alpha)\right)}{\partial\tau'}\frac{\partial x}{\partial t} + cos(\alpha)\frac{\partial}{\partial\tau'}\left(\frac{\partial x}{\partial t}\right) = \frac{c^2}{2}\left(cos(\alpha)\right)^8\frac{\partial\alpha}{\partial x}\left(\frac{-2}{\left(cos(\alpha)\right)^3}\left(-sin(\alpha)\right)\right)$$

$$cos(\alpha)\frac{\partial\alpha}{\partial t}\left(-sin(\alpha)\right) = -\frac{1}{2}\frac{\partial\alpha}{\partial t}\left(\frac{-2}{\left(cos(\alpha)\right)^3}\left(-sin(\alpha)\right)\right)\left(cos(\alpha)\right)^4$$
$$\frac{\partial t}{\partial\tau'}\frac{\partial\alpha}{\partial t}\frac{\partial\left(cos(\alpha)\right)}{\partial\alpha}\frac{\partial x}{\partial t} + cos(\alpha)\frac{\partial t}{\partial\tau'}\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial t}\right) = c^2\left(cos(\alpha)\right)^5 sin(\alpha)\frac{\partial\alpha}{\partial x}$$

$$-cos(\alpha)sin(\alpha)\frac{\partial\alpha}{\partial t} = -cos(\alpha)sin(\alpha)\frac{\partial\alpha}{\partial t}$$
$$-cos(\alpha)sin(\alpha)\frac{\partial\alpha}{\partial t}\frac{\partial x}{\partial t} + cos^2(\alpha)\frac{\partial^2 x}{\partial t^2} = c^2 cos^5(\alpha)sin(\alpha)\frac{\partial\alpha}{\partial x}$$

$$1 = 1$$
$$\frac{\partial^2 x}{\partial t^2} = c^2 cos^3(\alpha)sin(\alpha)\frac{\partial\alpha}{\partial x} + tan(\alpha)\frac{\partial\alpha}{\partial t}\frac{\partial x}{\partial t}$$

This first equation is trivial.

Let's work only the second equation. There is $\frac{\partial x}{\partial t}=0$ therefore:

$$\frac{\partial^2 x}{\partial t^2}=c^2 cos^3(\alpha)sin(\alpha)\frac{\partial\alpha}{\partial x}$$
$$\frac{\partial^2 x}{\partial t^2}=c^2 cos^4(\alpha)tan(\alpha)\frac{\partial\alpha}{\partial x}$$
$$\boxed{\frac{\partial^2 x}{\partial t^2}=c^2 cos^4(\alpha)tan(\alpha)\frac{\partial\, tan(\alpha)}{\partial x}} \qquad \textbf{(14)}$$

Substituting $cos(\alpha)$ by $\frac{\sqrt{1+e}}{1+\frac{e}{2}}$ (postulate 3), with $e=\frac{\sqrt{\frac{8R}{x}}}{s}$ (galaxy model), and $R=\frac{M_1 G}{c^2}$, where $M_1$ is the attracting mass, $G$ the gravitational constant, and c, light speed, we get :

$tan^2(\alpha)=\frac{e^2}{4(1+e)}$, and

$\frac{\partial^2 x}{\partial t^2}=c^2 cos^4(\alpha)tan(\alpha)\frac{\partial\, tan(\alpha)}{\partial x}$ yields :

$$\frac{\partial^2 x}{\partial t^2}=c^2 cos^4(\alpha)\frac{1}{2}\frac{\partial\left(tan^2(\alpha)\right)}{\partial x}$$
$$\frac{\partial^2 x}{\partial t^2}=c^2\left(\frac{\sqrt{1+e}}{1+\frac{e}{2}}\right)^4\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{e^2}{4(1+e)}\right)$$
$$\frac{\partial^2 x}{\partial t^2}=\frac{c^2}{8}\frac{(1+e)^2}{\left(1+\frac{e}{2}\right)^4}\frac{\partial}{\partial x}\left(\frac{e^2}{1+e}\right)$$
$$\frac{\partial^2 x}{\partial t^2}=\frac{c^2}{8}\frac{(1+e)^2}{\left(1+\frac{e}{2}\right)^4}\frac{2e(1+e)-e^2}{(1+e)^2}\frac{\partial e}{\partial x}$$
$$\frac{\partial^2 x}{\partial t^2}=\frac{c^2}{4}\frac{(1+e)^2}{\left(1+\frac{e}{2}\right)^4}\frac{e\left(1+\frac{e}{2}\right)}{(1+e)^2}\frac{\partial e}{\partial x}$$
$$\boxed{\frac{\partial^2 x}{\partial t^2}=\frac{c^2}{4}\frac{e}{\left(1+\frac{e}{2}\right)^3}\frac{\partial e}{\partial x}} \qquad \textbf{(15)}$$

Now substituting with $e=\frac{\sqrt{\frac{8R}{x}}}{s}$, we get

$$\frac{\partial e}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\sqrt{\frac{8R}{x}}}{s}\right)=\frac{\frac{\partial}{\partial x}\left(\sqrt{\frac{8R}{x}}\right)s-\sqrt{\frac{8R}{x}}\frac{\partial s}{\partial x}}{s^2}=\frac{\frac{1}{2}\sqrt{\frac{x}{8R}}\left(\frac{-8R}{x^2}\right)s-\sqrt{\frac{8R}{x}}\frac{\partial s}{\partial x}}{s^2}$$

:

$$=\frac{-\frac{\sqrt{2R}}{x^{\frac{3}{2}}}s-\sqrt{\frac{8R}{x}}\frac{\partial s}{\partial x}}{s^2}=-\frac{\sqrt{2R}}{x^{\frac{3}{2}}}\frac{s+2x\frac{\partial s}{\partial x}}{s^2}$$

And (15) becomes :

$$\frac{\partial^2 x}{\partial t^2}=\frac{c^2}{4}\frac{e}{\left(1+\frac{e}{2}\right)^3}\frac{\partial e}{\partial x}$$

$$\frac{\partial^2 x}{\partial t^2}=-\frac{c^2}{4}\frac{\sqrt{2R}}{x_i^{\frac{3}{2}}}\frac{\frac{\sqrt{\frac{8R}{x}}}{s}}{\left(1+\frac{1}{2}\frac{\sqrt{\frac{8R}{x}}}{s}\right)^3}\frac{s+2x\frac{\partial s}{\partial x}}{s^2}$$

$$\frac{\partial^2 x}{\partial t^2}=-\frac{Rc^2}{x_i^2}\frac{s+2x\frac{\partial s}{\partial x}}{\left(s+\sqrt{\frac{2R}{x}}\right)^3}$$

$$\boxed{\frac{\partial^2 x}{\partial t^2}=-\frac{Rc^2}{x_i^2}\frac{s+2x\frac{\partial s}{\partial x}}{\left(s+\sqrt{\frac{2R}{x}}\right)^3}}\qquad\textbf{(16)}$$

**<u>"Galactic" case, with *s* != 1, and x>>R :</u>**

$$\frac{\partial^2 x}{\partial t^2}=-\frac{M_1G}{x^2}\frac{s+2x\frac{\partial s}{\partial x}}{s^3}$$

with s = 1+f = 1+r/x for instance, which gives as usual:

$$\frac{\partial^2 x}{\partial t^2}=-\frac{M_1G}{x^2}\frac{1+\frac{r}{x}+2x\frac{\partial}{\partial x}\left(\frac{r}{x}\right)}{\left(1+\frac{r}{x}\right)^3}$$

$$\frac{\partial^2 x}{\partial t^2}=-\frac{M_1G}{x^2}\frac{1+\frac{r}{x}-2\frac{r}{x}}{\left(1+\frac{r}{x}\right)^3}$$

$$\frac{\partial^2 x}{\partial t^2}=-M_1G\frac{x-r}{(x+r)^3}$$

**Approximation with *s*=1, and x>>R :**

$$\frac{\partial^2 x}{\partial t^2} = -\frac{Rc^2}{x_i^2}\left(1+\sqrt{\frac{2R}{x}}\right)^{-3}$$

$$\frac{\partial^2 x}{\partial t^2} \simeq -\frac{Rc^2}{x_i^2}\left(1-3\sqrt{\frac{2R}{x}}+\frac{(-3)(-4)}{2!}\frac{2R}{x}\right)$$

$$\frac{\partial^2 x}{\partial t^2} \simeq -\frac{Rc^2}{x_i^2}\left(1-3\sqrt{\frac{2R}{x}}+\frac{12R}{x}\right)$$

$$\frac{\partial^2 x}{\partial t^2} \simeq -\frac{M_1 G}{x^2}\left(1-3\sqrt{\frac{2R}{x}}+\frac{12R}{x}\right)$$

$$\boxed{\frac{\partial^2 x}{\partial t^2} \simeq -\frac{M_1 G}{x^2}\left(1-3\sqrt{\frac{2R}{x}}+\frac{12R}{x}\right)} \qquad \textbf{(17)}$$

**Approximation of the Minkowskian coefficient $g_{00}$ with *s*=1, and x>>R :**

$$g_{00} = cos^2(\alpha)$$

$$g_{00} = \left(\frac{\sqrt{1+e}}{1+\frac{e}{2}}\right)^2 = \frac{1+e}{\left(1+\frac{e}{2}\right)^2}$$

$$g_{00} = (1+e)\left(1+\frac{e}{2}\right)^{-2} \simeq (1+e)\left(1-2\frac{e}{2}+\frac{(-2)(-3)}{2!}\left(\frac{e}{2}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{e}{2}\right)^3\right)$$

$$g_{00} \simeq (1+e)\left(1-e+\frac{3e^2}{4}-\frac{e^3}{2}\right)$$

$$g_{00} \simeq 1-e+\frac{3e^2}{4}-\frac{e^3}{2}+e-e^2+\frac{3e^3}{4}$$

$$g_{00} \simeq 1-\frac{e^2}{4}+\frac{e^3}{4}$$

$$g_{00} \simeq 1-\frac{1}{4}\left(\sqrt{\frac{8R}{x}}\right)^2+\frac{1}{4}\left(\sqrt{\frac{8R}{x}}\right)^3$$ with $R=\frac{M_1 G}{c^2}$, and $M_1$ being the attracting mass.

$$g_{00} \simeq 1-\frac{2R}{x}+\frac{2R}{x}\sqrt{\frac{8R}{x}}$$

$$g_{00} \simeq 1-\frac{2R}{x}\left(1-\sqrt{\frac{8R}{x}}\right)$$

$$g_{00} \simeq 1-\frac{M}{x}\left(1-2\sqrt{\frac{M}{x}}\right)$$ with $M=2R=\frac{2M_1 G}{c^2}$ being the Schwartzchild ray.

$$\boxed{g_{00} \simeq 1-\frac{2M_1 G}{c^2 x}\left(1-\frac{2}{c}\sqrt{\frac{2M_1 G}{x}}\right)} \qquad \textbf{(18)}$$

As calculated in the "Pioneer anomaly document" (PioneerExpl.doc).

# 5. EXACT RESULTS

This describes other possible equations related to the "Gravitational Model of the Three Elements Theory," *Journal of Modern Physics*, Vol. 3 No. 5, 2012, pp. 388-397.

$$F = mc^2 \frac{\mathrm{d}\tan(\alpha)}{\mathrm{dx}} \frac{\tan(\alpha)}{(1-\tan^2(\alpha))^{3/2}} \qquad \textbf{(19)}$$

could be replaced by :

$$F = mc^2\cos^4(\alpha)\tan(\alpha)\frac{d\tan(\alpha)}{dx} \qquad \textbf{(20)}$$

and

$$F = -\frac{mMG'}{x^2} \frac{\left(s+\sqrt{\frac{2R'}{x}}\right)\left(s+2x\frac{ds}{dx}\right)}{\sqrt{s^2+s\sqrt{\frac{8R'}{x}}}\left(s^2+s\sqrt{\frac{8R'}{x}}-\frac{2R'}{x}\right)^{3/2}} \qquad \textbf{(21)}$$

could be replaced by :

$$F = -\frac{mMG'}{x^2}\frac{s+2x\frac{\partial s}{\partial x}}{\left(s+\sqrt{\frac{2R'}{x}}\right)^3} \qquad \textbf{(22)}$$

But this would have no incidence in the final overall result.
(Of course, now above *M* is the attracting mass, and *G'* is the gravitational constant valid infinitely far from the attracting mass, therefore *G'* value is very close to *G* value in usual cases). Also :

| Planet | GR value | 3elt value |
|---|---|---|
| **Mercury** | 42.7848 | GR value – 0.0014 |
| **Saturn** | 1.66291 | GR value + 0.00056 |

should be replaced by :

| Planet | GR value | 3elt value |
|---|---|---|
| **Mercury** | 42.7848 | GR value + 0.0044 |
| **Saturn** | 1.66291 | GR value + 0.00061 |

But this has no incidence on the overall result which is the fact that those differences aren't strong enough, taking in mind that the model is not correct (solar system is not constaly filled with matter and therefore anyway those calculations must be re-executed using a varying matter density).

**Relative erratum error :**

$$Err\left(\frac{\partial^2 x}{\partial t^2}\right)^{-1} = \cos^4(\alpha)\left(1-\tan^2(\alpha)\right)^{\frac{3}{2}}$$ Equation (20) over equation (19).

$$Err\left(\frac{\partial^2 x}{\partial t^2}\right)^{-1} \simeq \left(1-\frac{R}{x}\right)^4\left(1-\frac{2R}{x}\right)^{\frac{3}{2}}$$

$$Err\left(\frac{\partial^2 x}{\partial t^2}\right)^{-1} \simeq \left(1-\frac{4R}{x}\right)\left(1-\frac{3R}{x}\right)$$

$$Err\left(\frac{\partial^2 x}{\partial t^2}\right)^{-1} \simeq 1-\frac{7R}{x}$$

This check that equation (17) is correct, because it is equation $\frac{\partial^2 x}{\partial t^2} \simeq -\frac{M_1 G}{x^2}\left(1-3\sqrt{\frac{2R}{x}}+\frac{19R}{x}\right)$, available in the document mystmass.pdf, (equation number (17)), but to its right factor must be added this value $-\frac{7R}{x}$.

The final result can be written

$$\boxed{MinkowskianOverMystmassdoc\,RelativeError \simeq 1-\frac{7R}{x}=1-\frac{7M_1G}{c^2x}} \qquad \textbf{(23)}$$

Now using equations (12), and (13), if we write the right term of last equation (12) over the right term of last equation (13), there is also :

$$MinkowskianOverRiemmannian\,RelativeError = h_{11}^4 = \left(1-\frac{M}{r}\right)^4 = \left(cos^2(\alpha)\right)^4 = \left(1-\frac{R}{x}\right)^8 \simeq 1-\frac{8R}{x}$$

$$MinkowskianOverRiemmannian\,RelativeError \simeq 1-\frac{8R}{x}=1-\frac{8M_1G}{c^2x}$$

And finally :

$$Err\left(\frac{\partial^2 x}{\partial t^2}\right)^{-1} = \cos^{-4}(\alpha)\left(1-\tan^2(\alpha)\right)^{\frac{3}{2}}$$ Last equation (13) over equation (19).

$$Err\left(\frac{\partial^2 x}{\partial t^2}\right)^{-1} = \left(1-\frac{R}{x}\right)^{-4}\left(1-\frac{2R}{x}\right)^{3/2}$$

$$Err\left(\frac{\partial^2 x}{\partial t^2}\right)^{-1} \simeq \left(1+\frac{4R}{x}\right)\left(1-\frac{3R}{x}\right)$$

$$Err\left(\frac{\partial^2 x}{\partial t^2}\right)^{-1} \simeq 1+\frac{R}{x}$$

$$RiemmannianOverMystmassdoc\,RelativeError \simeq 1-\frac{R}{x}=1-\frac{M_1G}{c^2x} \qquad \textbf{(24)}$$

Intermediate conclusion about the limited developments :

Riemmann → 1-R/x → Mystmass → 1-7R/x → Minkowski.

| Version | Equation | Equation with the slope angle |
|---|---|---|
| **Minkowski (correct one)** | $F=-\frac{mM_1G'}{x^2}\frac{s+2x\frac{\partial s}{\partial x}}{\left(s+\sqrt{\frac{2R}{x}}\right)^3}$ | $F = mc^2\cos^4(\alpha)\tan(\alpha)\frac{d\tan(\alpha)}{dx}$ |
| **Old one "mystmass.doc"** | $F=-\frac{mM_1G'}{x^2}\frac{\left(s+\sqrt{\frac{2R'}{x}}\right)\left(s+2x\frac{ds}{dx}\right)}{\sqrt{s^2+s\sqrt{\frac{8R'}{x}}}\left(s^2+s\sqrt{\frac{8R'}{x}}-\frac{2R'}{x}\right)^{3/2}}$ | $F = mc^2\frac{d\tan(\alpha)}{dx}\frac{\tan(\alpha)}{(1-\tan^2(\alpha))^{3/2}}$ |
| **Riemannian time-line geodesic** | Not interesting | $F = mc^2\cos^{-4}(\alpha)\tan(\alpha)\frac{d\tan(\alpha)}{dx}$ |

| Version | Approximation $s\simeq 1$ | Approximation $x \gg R'$ |
|---|---|---|
| **Minkowski (correct one)** | $F=-\frac{mM_1G'}{x^2}\frac{1}{\left(1+\sqrt{\frac{2R'}{x}}\right)^3}$ | $F=-\frac{mM_1G'}{x^2}\frac{s+2x\frac{\partial s}{\partial x}}{s^3}$ |
| **Old one "mystmass.doc"** | $F=-\frac{mM_1G'}{x^2}\frac{1+\sqrt{\frac{2R'}{x}}}{\sqrt{1+\sqrt{\frac{8R'}{x}}}\left(1+\sqrt{\frac{8R'}{x}}-\frac{2R'}{x}\right)^{3/2}}$ | $F=-\frac{mM_1G'}{x^2}\frac{s+2x\frac{ds}{dx}}{s^2\sqrt{s^2+s}}$ |
| **Riemannian time-line geodesic** | Not interesting | Not interesting |

| Version | Limited development |
|---|---|
| **Minkowski (correct one)** | $\frac{\partial^2 x}{\partial\tau^2}\simeq-\frac{M_1G'}{x^2}\left(1-3\sqrt{\frac{2R'}{x}}+\frac{12R'}{x}\right)$ |
| **Old one "mystmass.doc"** | $\frac{\partial^2 x}{\partial\tau^2}\simeq-\frac{M_1G'}{x^2}\left(1-3\sqrt{\frac{2R'}{x}}+\frac{19R'}{x}\right)$ |
| **Riemannian time-line geodesic** | $\frac{\partial^2 x}{\partial\tau^2}\simeq-\frac{M_1G'}{x^2}\left(1-3\sqrt{\frac{2R'}{x}}+\frac{20R'}{x}\right)$ |

$$R'=\frac{M_1G'}{c^2}=\frac{M'}{2}$$

*G'* is the gravitational constant valid infinitely far from the attracting mass, therefore *G'* value is very close to *G* value in usual cases.

## WRITING GEODESIC EQUATIONS WITH MINKOWSKIAN COEFFICIENTS

That's the more natural way to calculate. It will check the above results. Starting from equation (2) :

$$\frac{\partial^2 x^0}{\partial \tau^2} = -\frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^0} \left( \frac{\partial x^0}{\partial \tau} \right)^2$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = \frac{1}{2} g^{11} \frac{\partial g_{00}}{\partial x^1} \left( \frac{\partial x^0}{\partial \tau} \right)^2$$

Since $g^{11} = \frac{1}{g_{11}} = -g_{00}$ , last equation can be written :

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{1}{2} g_{00} \frac{\partial g_{00}}{\partial x^1} \left( \frac{\partial x^0}{\partial \tau} \right)^2$$

Now it must be used also, because the trajectory is a time line : $\frac{\partial x^0}{\partial \tau} = \frac{\partial ct}{\partial \tau} = c \frac{\partial t}{\partial \tau} = c \frac{dt}{\sqrt{g_{00}} dt} = \frac{c}{\sqrt{g_{00}}}$ . Therefore there is :

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{1}{2} g_{00} \frac{\partial g_{00}}{\partial x^1} \left( \frac{c}{\sqrt{g_{00}}} \right)^2 = -\frac{1}{2} g_{00} \frac{\partial g_{00}}{\partial x^1} \frac{c^2}{g_{00}} = -\frac{c^2}{2} \frac{\partial g_{00}}{\partial x^1}$$

And equations (2) are also finally :

$$\frac{\partial^2 x^0}{\partial \tau^2} = -\frac{c^2}{2} g_{00}^{-2} \frac{\partial g_{00}}{\partial x^0} \qquad \textbf{(25)}$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{c^2}{2} \frac{\partial g_{00}}{\partial x^1}$$

Which is Minkowskian time line geodesic equations.

Now let's restart from equations (3):

$$\frac{\partial^2 x^0}{\partial \tau'^2} = -\frac{1}{2} h^{00} \frac{\partial h_{00}}{\partial x^0} \left( \frac{\partial x^0}{\partial \tau'} \right)^2$$

$$\frac{\partial^2 x^1}{\partial \tau'^2} = \frac{1}{2} h^{11} \frac{\partial h_{00}}{\partial x^1} \left( \frac{\partial x^0}{\partial \tau'} \right)^2$$

There is :

$$\frac{\partial^2 x^0}{\partial \tau'^2} = \frac{\partial \tau}{\partial \tau'} \frac{\partial}{\partial \tau} \left( \frac{\partial \tau}{\partial \tau'} \frac{\partial x^0}{\partial \tau} \right) = g_{00} \frac{\partial}{\partial \tau} \left( g_{00} \frac{\partial x^0}{\partial \tau} \right) = g_{00} \frac{\partial g_{00}}{\partial \tau} \frac{\partial x^0}{\partial \tau} + g_{00}^2 \frac{\partial^2 x^0}{\partial \tau^2}$$

Therefore first equation of (3) is also the following.

$$g_{00} \frac{\partial g_{00}}{\partial \tau} \frac{\partial x^0}{\partial \tau} + g_{00}^2 \frac{\partial^2 x^0}{\partial \tau^2} = -\frac{1}{2} g_{00} \frac{\partial}{\partial x^0} \left( \frac{1}{g_{00}} \right) \left( \frac{\partial x^0}{\partial \tau'} \right)^2$$

$$\frac{\partial^2 x^0}{\partial \tau^2} = -g_{00}^{-1} \frac{\partial g_{00}}{\partial \tau} \frac{\partial x^0}{\partial \tau} + \frac{1}{2} g_{00}^3 \frac{1}{g_{00}^2} \frac{\partial g_{00}}{\partial x^0} \left( \frac{\partial \tau}{\partial \tau'} \frac{\partial x^0}{\partial \tau} \right)^2$$

$$\frac{\partial^2 x^0}{\partial \tau^2} = -g_{00}^{-1} \frac{\partial g_{00}}{\partial x^0} \frac{\partial x^0}{\partial \tau} \frac{\partial x^0}{\partial \tau} + \frac{1}{2} g_{00} \frac{1}{g_{00}^2} \frac{\partial g_{00}}{\partial x^0} \left( \frac{\partial x^0}{\partial \tau} \right)^2$$

$$\frac{\partial^2 x^0}{\partial \tau^2} = -g_{00}^{-1} \frac{\partial g_{00}}{\partial x^0} \left( \frac{\partial x^0}{\partial \tau} \right)^2 + \frac{1}{2} g_{00}^{-1} \frac{\partial g_{00}}{\partial x^0} \left( \frac{\partial x^0}{\partial \tau} \right)^2$$

$$\frac{\partial^2 x^0}{\partial \tau^2} = -\frac{1}{2} g_{00}^{-1} \frac{\partial g_{00}}{\partial x^0} \left( \frac{\partial x^0}{\partial \tau} \right)^2$$

There is :

$$\frac{\partial^2 x^1}{\partial \tau'^2} = \frac{\partial \tau}{\partial \tau'} \frac{\partial}{\partial \tau} \left( \frac{\partial \tau}{\partial \tau'} \frac{\partial x^1}{\partial \tau} \right) = g_{00} \frac{\partial}{\partial \tau} \left( g_{00} \frac{\partial x^1}{\partial \tau} \right) = g_{00} \frac{\partial g_{00}}{\partial \tau} \frac{\partial x^1}{\partial \tau} + g_{00}^2 \frac{\partial^2 x^1}{\partial \tau^2} = g_{00}^2 \frac{\partial^2 x^1}{\partial \tau^2}$$

Therefore last equation of (3) is also the following.

$$\frac{\partial^2 x^1}{\partial \tau'^2} = \frac{1}{2} h^{11} \frac{\partial h_{00}}{\partial x^1} \left( \frac{\partial x^0}{\partial \tau'} \right)^2$$

$$g_{00}^2 \frac{\partial^2 x^1}{\partial \tau^2} = \frac{1}{2} \frac{1}{g_{00}} \frac{\partial}{\partial x^1} \left( \frac{1}{g_{00}} \right) \left( \frac{\partial \tau}{\partial \tau'} \frac{\partial x^0}{\partial \tau} \right)^2$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{1}{2} g_{00}^{-3} \frac{1}{g_{00}^2} \frac{\partial g_{00}}{\partial x^1} \left( g_{00} \frac{\partial x^0}{\partial \tau} \right)^2$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{1}{2} g_{00}^{-3} \frac{\partial g_{00}}{\partial x^1} \left( \frac{\partial x^0}{\partial \tau} \right)^2$$

Now it must be used also, because the trajectory is a time line : $\frac{\partial x^0}{\partial \tau} = \frac{\partial ct}{\partial \tau} = c \frac{\partial t}{\partial \tau} = c \frac{dt}{\sqrt{g_{00}}\, dt} = \frac{c}{\sqrt{g_{00}}}$. Therefore there is :

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{1}{2} g_{00}^{-3} \frac{\partial g_{00}}{\partial x^1} \left( \frac{c}{\sqrt{g_{00}}} \right)^2 = -\frac{1}{2} g_{00}^{-3} \frac{\partial g_{00}}{\partial x^1} \frac{c^2}{g_{00}} = -\frac{c^2}{2} g_{00}^{-4} \frac{\partial g_{00}}{\partial x^1}$$

And equations (3) are also finally :

$$\boxed{\begin{aligned} \frac{\partial^2 x^0}{\partial \tau^2} &= -\frac{c^2}{2} g_{00}^{-2} \frac{\partial g_{00}}{\partial x^0} \qquad \textbf{(26)} \\ \frac{\partial^2 x^1}{\partial \tau^2} &= -\frac{c^2}{2} g_{00}^{-4} \frac{\partial g_{00}}{\partial x^1} \end{aligned}}$$

Which is Riemannian time line geodesic equations, written using Minkowskian coefficients and local time for comparison.

Now the comparison is between equations (25) and (26) and is a little bit easier to understand (with Minlowskian variables).

Let's verify the calculations above with this new formulation. That's the Riemannian trajectory:

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{c^2}{2} g_{00}^{-4} \frac{\partial g_{00}}{\partial x^1}$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{c^2}{2}\left((cos(\alpha))^2\right)^{-4} \frac{\partial}{\partial x}\left((cos(\alpha))^2\right)$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{c^2}{2}\left(\frac{\sqrt{1+e}}{1+\frac{e}{2}}\right)^{-8} \frac{\partial}{\partial x}\left(\frac{1+e}{\left(1+\frac{e}{2}\right)^2}\right)$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{c^2}{2}\frac{\left(1+\frac{e}{2}\right)^8}{(1+e)^4}\frac{\left(1+\frac{e}{2}\right)^2-(1+e)2\left(1+\frac{e}{2}\right)\frac{1}{2}}{\left(1+\frac{e}{2}\right)^4}\frac{\partial e}{\partial x}$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{c^2}{2}\left(1+\frac{e}{2}\right)^4\frac{\left(1+e+\frac{e^2}{4}\right)-\left(1+\frac{3e}{2}+\frac{e^2}{2}\right)}{(1+e)^4}\frac{\partial e}{\partial x}$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = \frac{c^2}{4}\frac{\left(1+\frac{e}{2}\right)^5}{(1+e)^4}e\frac{\partial e}{\partial x}$$

Now using $\frac{\partial e}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{\frac{8R}{x}}\right) = \frac{1}{2e}\left(-\frac{8R}{x^2}\right)$ :

$$\frac{\partial^2 x^1}{\partial \tau^2} = \frac{c^2}{4}\frac{\left(1+\frac{e}{2}\right)^5}{(1+e)^4}e\frac{1}{2e}\left(-\frac{8R}{x^2}\right)$$

$$\frac{\partial^2 x}{\partial \tau^2} = -\frac{Rc^2}{x^2}\frac{\left(1+\frac{e}{2}\right)^5}{(1+e)^4}$$

Limited development:

$$\frac{\partial^2 x}{\partial \tau^2} \simeq -\frac{Rc^2}{x^2}\left(1+\frac{5e}{2}+\frac{(5)(4)}{2!}e^2/4\right)\left(1-4e+\frac{(-4)(-5)}{2!}e^2\right)$$

$$\frac{\partial^2 x}{\partial \tau^2} \simeq -\frac{Rc^2}{x^2}\left(1+\frac{5e}{2}+\frac{5e^2}{2}\right)\left(1-4e+10e^2\right)$$

$$\frac{\partial^2 x}{\partial \tau^2} \simeq -\frac{Rc^2}{x^2}\left(1-4e+10e^2+\frac{5e}{2}+\frac{5e}{2}(-4e)+\frac{5e^2}{2}\right)$$

$$\frac{\partial^2 x}{\partial \tau^2} \simeq -\frac{Rc^2}{x^2}\left(1-\frac{3e}{2}+\frac{5e^2}{2}\right)$$

$$\frac{\partial^2 x}{\partial \tau^2} \simeq -\frac{Rc^2}{x^2}\left(1-\frac{3e}{2}+\frac{5e^2}{2}\right)$$

$$\frac{\partial^2 x}{\partial \tau^2} \simeq -\frac{M_1 G}{x^2}\left(1-\frac{3}{2}\sqrt{\frac{8R}{x}}+\frac{5}{2}\left(\sqrt{\frac{8R}{x}}\right)^2\right)$$

$$\frac{\partial^2 x}{\partial \tau^2} \simeq -\frac{M_1 G}{x^2}\left(1-3\sqrt{\frac{2R}{x}}+\frac{20R}{x}\right)$$

This is the same result as above for Riemannian version (20R/x = 12R/x + 7R/x + R/x). Therefore, equation (17) can also be retrieved from (25), and this is the most simple and straightforward way to calculate it.

And starting from the Minkowskian trajectory:

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{c^2}{2}\frac{\partial g_{00}}{\partial x^1}$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{c^2}{2}\frac{\partial}{\partial x}\left(\left(cos(\alpha)\right)^2\right)$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{c^2}{2}\frac{\partial}{\partial x}\left(\frac{1+e}{\left(1+\frac{e}{2}\right)^2}\right)$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{c^2}{2}\frac{\left(1+\frac{e}{2}\right)^2-(1+e)2\left(1+\frac{e}{2}\right)\frac{1}{2}}{\left(1+\frac{e}{2}\right)^4}\frac{\partial e}{\partial x}$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{c^2}{2}\frac{\left(1+e+\frac{e^2}{4}\right)-\left(1+\frac{3e}{2}+\frac{e^2}{2}\right)}{\left(1+\frac{e}{2}\right)^4}\frac{\partial e}{\partial x}$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = \frac{c^2}{4}\frac{\left(1+\frac{e}{2}\right)}{\left(1+\frac{e}{2}\right)^4}e\frac{\partial e}{\partial x}$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = \frac{c^2}{4}\frac{e}{\left(1+\frac{e}{2}\right)^3}\frac{\partial e}{\partial x}$$ As expected with (15).

Now using $\frac{\partial e}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{\frac{8R}{x}}\right) = \frac{1}{2e}\left(-\frac{8R}{x^2}\right)$ :

$$\frac{\partial^2 x^1}{\partial \tau^2} = \frac{c^2}{4}\frac{e}{\left(1+\frac{e}{2}\right)^3}\frac{1}{2e}\left(-\frac{8R}{x^2}\right)$$

$$\frac{\partial^2 x^1}{\partial \tau^2} = -\frac{Rc^2}{x^2}\frac{1}{\left(1+\frac{e}{2}\right)^3}$$

As expected with (16).

## 6. SCHWARTZCHILD METRIC IN THIS MODEL

Time dilatation between non-inertial and free falling particle trajectories is equal to cos(α), whith α being the slope angle of local space line with respect to the inertial reference frame attached to the universe. This uses the Riemannian representation of space-time. In this representation the free falling particle trajectory is a time line (but not a Riemannian geodesic) and there is also tan(α) = v/c. Postulate 3 tells that cos(α) = oper(L1,L2), and therefore it yields the metric in any locally uniformly filled space case : g00 = (dτ/dt)² = cos²(α) = oper(L1,L2)². In the specific case of a unique attracting object in a universe filled with a constant density of matter, this gives the Swartzchild time dilation (1+e)/(1+e/2)² with e being $2\sqrt{M/x}$ and M the Schwarzchild ray. Limited development : g00 = (1+e)/(1+e/2)² ≅ (1+e) (1-e+(-2)(-3)/2 e²/4 + (-2)(-3)(-4)/6 e3/8 + o(e3)) = (1+e) (1 - e + 3e²/4 - e3/2 + o(e3)) = 1 - e + 3e²/4 - e3/2 + o(e3) + e – e² + 3e3/4 + o(e3)) = 1 - e + 3/4e² - e3/2 + e –e² + 3e3/4 + o(e3) = 1 – e²/4 + e3/4 + o(e3) = 1 – 2R/x + 4√2(R/x)(3/2) + o(e3) = 1 – M/x + 2(M/x)√(M/x) + o(e3).

This must be compared with the result of the calculations from another calculation in the document "PioneerExpl.pdf :

$$g_{00} \simeq 1 - \frac{M'}{r}\left(1 - 2\sqrt{\frac{M'}{r}}\right) \qquad \text{and therefore} \qquad g_{00} \simeq 1 - \frac{M'}{r} + 2\frac{M'}{r}\sqrt{\frac{M'}{r}}$$

With $M' = \frac{2M_0G'}{c^2}$ being the Schwarzchild ray, $M_0$ being the attracting mass, and $G'$ the gravitational constant valid only for long distances approximatively equal to $G$ ).

Therefore the Schwarzchild metric in the context of the gravitational model of the three elements theory is the following.

$$ds^2 = \frac{1+2\sqrt{\frac{M'}{r}}}{\left(1+\sqrt{\frac{M'}{r}}\right)^2} c^2dt^2 - \frac{\left(1+\sqrt{\frac{M'}{r}}\right)^2}{1+2\sqrt{\frac{M'}{r}}} dr^2$$

(Spherical θ and φ variables are not indicated in the right hand term of this equation).
Of course this is not compatible with General Relativity tensorial curvature equation, since Newton's law is no longer correct now.
Close to the attracting object, time dilation is equal to dτ/dt = cos(α) = oper(L1,L2) = √(1+e)/(1+e/2) ≅ √e/(e/2) = 2/√e = 2/√√(8R/x) = √2 √√(x/M), so dτ/dt = √2 √√(x/M) and g00 ≅ <u>2√(x/M)</u>, which is of course a singularity but existing only for the mathematical limit x=0. In reality this singularity doesn't exist because it comes only from the unrealistic supposition of a point like massive object. Therefore there is no physical singularity, which is more realistic.

**<u>Remark</u>**

$$g_{00} \;=\; \frac{1+2\sqrt{\frac{M'}{r}}}{\left(1+\sqrt{\frac{M'}{r}}\right)^2} \;\simeq\; 1 - \frac{M'}{r} \qquad for\;\; r \gg M' \,but\; only\;\; for\; r \gg M'$$

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