Double integral region grapher

This work was partially supported by the National Science Foundation under grant DUE-0535327. You can download the complete source code for applet from the article Visualizing Regions for Double Integrals by the authors published on the MathDL Flash Forum at Digital Classroom Resources of MathDL. In this mathlet we use classes by Barbara Kaskosz and Doug Ensley published in the article Flash Tools for Developers: Parametric Curves on the Plane at MathDL Flash Forum. Applets' Home Kaskosz Home Disclaimer Show Mobile Notice Show All Notes Hide All Notes Mobile Notice You appear to be on a device with a narrow screen width (i.e. you're probably on a mobile phone). Due to the nature of mathematics on this site is the best view in landscape mode. If your device is not in landscape mode, many of the equations will run out from the side of the device (should be able to scroll to see them), and some of the menu items will be cut off due to the narrow screen width. In the previous section we looked at double integrals over rectangular regions. The problem with this is that most areas are not rectangular, so we now have to look at the following double integral, \[\iint\limits_{D}{{f\left({x,y} \right)\,dA}}\] where \(D\) is any range. There are two types of regions that we need to look at. Here's a sketch of both of them. We will often use set builder notation to describe these regions. Here is the definition of the range in case 1 \[D = \left\{ {\left({x,y} \right)|a \le x \le b,\,\,{g_1}\left(x \right) \le y \le {g_2}\left(x \right)} \right\}\] and here is the definition for the area in case 2. \[D= \left\{ {\left({x,y} \right)|{ h_1}\left(y \right) \le x \le {h_2}\left(y \right),\,c \le y \le d} \right\}\] This notation is really just a smart way to say, that we need all the points\ (\left({x,y} \right)\), where both coordinates meet the two given inequalities. The dual integral for both of these cases is defined in terms of iterated integrals as follows. In case 1 where \(D = \left\{ {\left({x,y} \right)|a \le x \le b,\\\,{g_1}\left(x \right) \le y \le {g_2}\left(x \right)} \right\}\) is defined as, the integral is defined to be, \[\iint\limits_{D}{{f\left, {x,y} \right)\,dA}} = \int_{{\,a}}^{{,b}{{{\int_{{{g_{\,1}\left(x \right)}}^{{{{g_{\,2}\left(x \right)}}{f\left({x .y} \right)\ ,dy}}\,dx}}\] In Case 2, where \(D= \left\{ {\left({x,y} \right)|{ h_1}\left(y \right) \le x \le {h_2}\left(y \right),\,c \le y \le d} \right\}\) Integral is defined to be\[\iint\limits_{D}{{f\left({x,y} \right)\,dA}} = \int_{{\,c}{{{d}}{\T int_{{h{{,_1}\left y \right)}}{{f\left({x,y} \right)\,dx}}\,dy}}\] Here are some dual integral properties that we should review before we actually do any examples. Note that all three of these properties are just extensions of properties of single integrals that have been extended to double integrals. Properties \(\displaystyle \iint\limits_{D}{{f\left({x,y} \right) + g\left({x,y} \right)\,dA}} = \iint\limits_{D}{{f\left({x,y} \right)\,dA}} + \iint\limits_{D}{{g\left ({x,y} \right)\,dA}}\) \(\displaystyle \iint\limits_{d}{{cf\left({x,y} \right)\,dA}} = c\iint\limits_{D}{{f\left({x,y} \right)\,dA}}\), where \(c\) is constant. If the range \(D\) can be divided into two separate ranges \({D_1}\) and \({D_2}\), the integral can be written as \[\iint\limits_{D}{f\left\({x,y} \right)\,dA}} = \iint\limits_{{{{{{{D_1}}}{f\left, {x,y} \right)\,dA}} + \iint\limits_{{D_2}}}{f\left({x,y} \right)\,dA}}\] Let's see some examples of duplicate integrals over general ranges. Example 1 Evaluate each of the following integrals over the specified range \(D\). \(\displaystyle \iint\limits_{D}{{{{{{bf{e}}^{\frac{x}{y}}}\,dA}}\), \(D = \left\{ {\left\ {x,y} \right)|1 \le y \y le 2,\,\,y \le x \\le {y^3}} \right\}\) \(\displaystyle \iint\limits_{D}{4xy - {y^3}\,dA}}\), \(D\) is the range of bounded by \(y = \sqrt x \) and \(y = {x^3}\). \(\displaystyle \iint\limits_{D}{6{x^2} - 40y\,dA}}\), \(D\) is the triangle with vertexs \(\left({0.3} \right)\), \(\left({1.1} \right)\) and \(\left({5.3} \right)\). Show All Solutions Hide All Solutions a \(\displaystyle \iint\limits_{D}{{{{bf{e}}^{\frac{x}{y}}}\,dA}}\), \(D = \left\{ {\{\frac{x}{y}}}\,dA}}\), \(D = \left\{ {\{\frac{x}{y}}}\\), \ (D= \left\{ {\{{\frac{x}{y}}}\\), \(D= \left\{ {\{{frac{x}{y}}}\\), \(D= \left\{ {\{frac{x}{y}}}\\), \(D= \left\{ left({x,y} \right)|1 \le y \le 2,\,\,y \le x \le {y^3}} \right\}\) Show solution Okay, this first one is configured to just use the formula above, let's do it. \[\begin{align*}\iint\limits_{D}{{{{bf{e}}^{\frac{x}{y}}}\ ,dA}} & = \int_{{{,1}}^{{,2}}{\int_{{{{\{y}}^{y^3}}{{{{{{{{bf{e}}^{\frac{x}{y}}}\,dx}}\,dy}} = \int_{{\,1}}^{{,2}}{{left. {y\,{{{bf{e}}^{\frac{x}{y}}}} \right|_y^{y^3}}\,dy}}\\ & = \int_{{\,1}}^{{,2}}{{y\,{bf{e}}^{y^2}}} - y{{\bf bf{e}}^1}\,dy}}\\ & = \left. {\left({\frac{1}{2}{{\bf{e}}^{{y^2}}} - \frac{1}{2}{y^2}{{{bf{e}}^1}} \right)} \right|_1^2 = \frac{1}{2}{{\bf{e}}^4} - 2{{{{{{e}}^4} - 2{{{{{{{{bf{e}}^1\end{align*}\] b \(\displaystyle \iint\limits_{D}{4xy - {y^3}\,dA}}\), \(D\) is the range of bounded by \(y = \sqrt x \) and \(y = {x^3}\). Show solution In this case, we need to determine the two inequalities for \(x\) and \(y\) that we need to make the integral. The best way to do this is to graph the two curves. Here's a sketch. So from the sketch we can see that two inequalities are, \[0 \le x \le 1\hspace{0.25in}{x^3} \le y \le \sqrt x \] We can now do the integral, \[\begin{align*}\iint\limits_{D}{{4xy - {y^3}\,dA}} & = \int_{{\,0}}^{{,1}}{{{\int_{{{,{,{x^^3}}}^{{,\sqrt x }}{{4xy - {y^3}\,dy}}\,dx}}\\ & = \int_{{\,0}}^{{,1}}{{{left. {\left ing {2x{y^2} - \frac{1}{4}{y^4}} \right)} \right|_{x^3}^{sqrt x }\,dx}}\\ & = \int_ {{\,0}}^{{,1}}{{\frac{7}{4}{x^2} - 2{x^7} + & = \left. {\left. {\left. - \frac{1}{4}{x^8} + \frac{1}{52}}{x^{13}} \right)} \right|_0^1 = \frac{55}}{156}\end{align*}\] c \ (\displaystyle \iint\limits_{D }{6{x^2} - 40y\,dA}}\), \(D\) is the triangle with vertex \(\left({0.3} \right)\), \(\left({1.1} \right)\) and \(\left({5.3} \right)\). View solution We got even less information about the region this time. Let's start this by drawing the triangle. Since we have two points on each edge it is easy to get equations for each edge, and then we will leave it to you to check the equations. Now there are two ways to describe this region. If we use functions \(x\), as shown in the image, we will have to break the region up into two different pieces, since the lower function is different depending on the value of \(x\). In this case, the range will be given by \(D = {D_1} \cup {D_2}\) where ,\[\begin{align*}{D_1} & | \left\{ {\left({x,y} \right)|0 \le x \le 1,\\\\ - 2x + 3 \le y \le 3} \right\}\\ {D_2} & = \left\{ {\left({x,y} \right)|1 \le x \le 5,\\\\\\\\\\\\\\\frac{1}{2}x + \frac{1}{2} \le y \le 3} \right\}\end{align*}\] Note ,that \(\cup \) is the symbol of union, and it simply means that \(D\) is the area we get by combing the two regions. If we do that, then we will have to make two separate integrals, one for each of the regions. To avoid this, we could turn things around and solve the two equations for \(x\) to get, \[\begin{align*}y & = - 2x + 3\hspace{0.5in} \Rightarrow \hspace{0.5in}x = - \frac{1}{2}y + \frac{3}{2}\\ y & = \frac{1}{2}x + \frac{1}{2}\hspace{0.5in} \Rightarrow \hspace{0.5in}x = 2y - 1\end{align*}\] If we do this , we may notice that the same function is always on the right, and the same function is always on the left, and so the range is \[D = \left\{ {\left({x,y} \right)|\\, - \frac{1}{2}y + \frac{3}{2} \le x \le 2y - 1,\,\\,1 \le y \le 3} \right\}\] Writing the area in this form means that you perform a single integrated device instead of the two integrals We have to do it in other ways. Either way should give the same answer and so we can get an example in the notes on splitting a region up let's do both integrals. Workaround 1 \[\begin{align*}\iint\limits_{D}{6{x^2} - 40y\,dA}} & = \iint\limits_{{D_1}}{6{x^2} - 40y\,dA}} + \iint\limits_{{D_2}}{6{x^2} - 40y\,dA}}\\ & = \int_{{{0}}^{{\,1}}{{{int_{{\, - 2x + 3}}^{{,3}}{6{x^2} - 40y\,dy}}\,dx}} + \int_{{\,1}}^{{,5}}{\int_{{\,\frac{1}{2}x + \frac{1}{2}}^{{\,3}}{6{x^2} - 40y\ ,dy}}\,dx}}\\\amp; = \int_{{\ .0}}^{{,1}}{{left. {\left({6{x^2}y - 20{y^2}} \right)} \right|_{ - 2x + 3}^3\,dx}} + \int_{{\,1}^{{\,5}}{{left. {\left ing {6{x^2}y - 20{y^2}} \right)} \right|_{\frac{1}{2}x + \frac{1}{2}}^3\,dx}}\\ & = \int_{{\,0}}^{{,1}}{{{12{x^3} - 180 + 20{{\left({3 - 2x} \right)}^2}\,dx}} + \int_{{\,1}}^{{,5}}{{ - 3{x^3} + 15{x ^2} - 180 + 20{{\left({\frac{1}{2}x + \højre)}^2}\,dx}}\\ & = \venstre. {\left({3{x^4} - 180x - \frac{10}}{3}{{\left. \frac{10}}{3}{{\left. - 2x} \right)}^3}} \right)} \right|_0^1 + \left. {\left({ - \frac{3}{4}{x^4} + 5{x^3} - 180x + \frac{40}}{3}{{\left({\frac{1}{2}x + \\\frac{1}{2}} \right)}^3}} \right)} \right|_1^5\\ & = - \frac{935}}{3}\end{align*}\] It was a lot of work. Note, however, that after we made the first substitution, we didn't multiply everything out. The two square expressions can be easily integrated with a basic Calc I substitution and so we don't bother to multiply them out. We will do it at times to make some of these integrals a little easier. Solution 2This solution will be much less work as we will only make a single integral. \[\begin*}\iint\limits_{D}{6{x^2} - 40y\,dA}} &= \int_{1}^{3}{\int_{{ - \frac{1}{2}y + \frac{3}{2}}}^{2y - 1}}{6{x^2} - 40y\,dx}}\,dy}}\amp; = \int_{1}^{3}{{left. {\left({2{x^3} - 40xy} \right)} \right|_{ - \frac{1}{2}y + \frac{3}{2}}^{2y - 1}\,dy}}\\amp; = \int_{1}^{3}{100y - 100y - 1}\,dy}}\\ & = \int_{1}^{3}{100y - 100y - 1}\,dy}}\\amp; = \int_{1}^{3}{100y - 100y - 1}\,dy}}\\ & = \int_{1}^{3}{100y - 100y - 1}\,dy}}\\ & = \int_{1} {3} <4>^{3}{10100{y^2} + 2{{\left({2y - 1} \right)}^3 - 2{{\left({ - \frac{1}{2}y + \frac{3}{2}} \right)}^ 3\,dy}}\\& = \left. {\left, {50{y^2} - \frac{100}}{3}{y^3 + \frac{1}{4}{{\left, {2y - 1} \right)}^4} + {{\left({ - \\left frac{1}{2}y + \frac{3}{2}} \right)}^4}} \right)} \right|_1^3\\ & = \frac{935}}{3}\end{align*}\] So the numbers were a little messier, but apart from, that there was much less work for the same result. Please also note that we again do not dice the two terms as they are easier to handle using a Calc I substitution. As the last part of the previous example has shown us, we can integrate these integrals in either order (i.e. \(x\) followed by \(y\) or \(y\) followed by \(x\)), although one order will often be lighter than the other. In fact, there will be times when it will not even be possible to make the integral in one order, while it will be possible to make the integral in the second order. Also, don't forget the calculus I'm replacing. Students often just get busy and multiply everything after doing the integrated evaluation and end up missing a really simple Calculus I substitution that avoids the hassle of multiplying everything out. Calculus I replacements don't always show up, but when they do they almost always simplify the work for the rest of the problem. Let's see a few examples of these kinds of integrals. Example 2 Evaluate the following integrals by first reversing the integration order. \(\displaystyle \int_{{{,0}}^{{,3}}{{{int_{{{{{{x^2}}^{{,9}}{{{{x^3}{{bf{e}}^{{{0y^3}}}\,dy}\,dx}}\ \.com displaystyle \int_{{\,0}}^{{\,8}{{{int_{{\,\sqrt[3]{y}}}^{{,2}}{{{sqrt {{x^4} + 1} \,dx}}\,dy}}\) Show all solutions hide all solutions a \(\displaystyle View solution first, note that if we try to integrate with respect to \(y\) we can't do the integral because we would need a \({y^ 2}\) front exponential to make \(y\) integration. We hope that, if we change the order of integration, we will have an integral part of what we can do. Now, when we say that we will reverse the order of integration it means that we want to integrate with respect to \ (x\) first and then \(y\). Also note that we can't just exchange integrals, keep the original limits, and be done with it. This would not solve our original problem, and to integrate with respect to \(x\) we cannot have \(x\) 's in the limits of integrals. Even if we ignored that answer would not be a constant, as it should be. So let's see how we reverse the order of integration. The best way to reverse the order of integration is to first outline the region specified by the original integration boundaries. From the integral, we see that the inequalities that define this region are, \[\begin{array}{c}0 \le x \le 3\\ {x^2} \le y \le 9\end array}\] These inequalities tell us, that we want the range of \(y = {x^2}\) on the lower limit and \(y = 9\) on the upper limit between \(x = 0\) and \(x = 3\). Here's a sketch of this region. Since we want to integrate with respect to \(x\) first we need to determine the limits of \(x\) (probably in the form of \(y\)) and then get the limits of \(y\)'s. Here they are for this region. \[\begin{array}{c}0 \le x \le \sqrt y \\ 0 \le y \le 9\end{array}\] All horizontal lines drawn in this range start at \(x = 9\end{array}\] All horizontal lines, drawn in this range starts at \(x = 9\end{array}\] All horizontal lines drawn in this range, starts at \(x = 00\) and end at \(x = \sqrt y \) and then these are the boundaries of \(x\)'s and range \(y\)'s for the areas are 0 to 9. The integral, with the order reversed, is now, \[\int_{{\,0}}^{{,3}}{{int_{{\,{2}}^{{,9}}{{{x{{{bf{e}}^{y^3}}}\,dy}}\,dx}} = \int_{{{{{{{{{{{{{{{2 {\,0}}^{{,9}}{{\int_{{\,0}^{\,\sqrt y }}{{x^3}{{bf{e}}^{{{y^3}}}\,dx}}\,dy}}\] and note that we can perform the first integration with this order. We also hope that this will give us another integral that we can do. Here is the work for this integral. \[\begin*}\int_{{{,0}}^{{,3}}{{{int_{{{,{,{x^2}}^{{,9}}{{{{x^3}{{bf{e}}^{{y^3}}}\,dy}},dx}} = \int_{{\,0 }}^{{,9}}{{{int_{{{{,0}}^{{,\sqrt y }}{{x^3}{{bf{e}}^{{y^3}}}\,dx}}\,dy}}\\ & = \int_{{\,0}}^{{,9}}{{left. {\frac{1}{4}{x^4}{{bf{e}}^{{y^3}}} \right|_0^{\sqrt y }\,dy}}\amp; = \int_{{{\,0}}^{{\,9}}{{\frac{1}{4}{y^2}{{{bf{e}}^{{y^3}}}\,dy}}\\ & = \left. {\frac{1}{12}}{{bf{e}}^{{y^3}}} \right|_0^9\\ & = \frac{1}{12}\left({{{bf{e}}^{729}} - 1} \right)\end{align*}\] So, as we hoped, we were able to do the integral when we exchanged the order of integration. b \(\displaystyle \int_{{{,0}}^{{,8} {{{\int_{{\,\sqrt[3]{y}}}^{{,2}}{{{sq not {{x^4} + 1} \,dx}}\,dy}}\) View view As with the first integral, we cannot make this integral by integrating with respect to \(x\) first, so we hope that by changing the order of integration we will get which we can integrate. Here are the limits of the variables that we get from this integral. \[\begin{array}{c}\sqrt[3]{y} \le x \le 2\\ 0 \le y \le 8\end{array}\] and here is a sketch of this range. So if we change the order of integration, we will have the following limits. \[\begin{array}{c}0 \le x \le 2\\ 0 \le \le y \le {x^3}\end{array}\] The integral is then, \[\begin*}\int_{{\,0}^{{,8}{{{{int_{\,\sqrt[3]{y}}^{{.2}{{{{sqrt {{x^4} + 1} \,dx}}\,dy}} & = \int_{{\,0}}^{{,2}}{{\int_{{\,0}}^{{{{{{x^3}} }{{sqrt {{x^4} + 1} \,dy}}\,dx}}\\ & & = \int_{{\,0}}^{{,2}}{{{left. {y\sqrt {{x^4} + 1} } \right|_0^{{x^3}\,dx}}\amp; = \int_{{\,0}}^{{\,2}{{{x^3}\sq 2 {{x^4} + 1} \,dx}} = \frac{1}{6}\left({{{17}^{\frac{3}{2}}} - 1} \right)\end{align*}\] The last subject in this paragraph is two geometric interpretations of a double integral. The first interpretation is an extension of the idea that we used to develop the idea of a double integral in the first paragraph of this chapter. We did this by looking at the volume of the substance that was below the surface of the function \(z = f\left({x,y} \right)\) and above the rectangle \(R\) in planet \(xy\) plane. This idea can be extended to more general regions. The volume of the substance below the surface specified by \(z = f\left({x,y} \right)\) and above the range \(D\) of plane \(xy\)-, is specified by, \[V = \iint\limits_{D} {{f\left, {x,y} \right)\,dA}}\] Sample 3 Find the volume of the substance below the surface indicated by the surface \(z = 1 6xy + 200\) and is above the range in the \(xy\) plane range bounded by \(y = {x^2}\) and \(y = 8 - {x^2}\). View solution Here is the graph of the surface, and we have tried to show the area in the area \(xy\) plane below the surface. Here is a sketch of the region in the \(xy\) plane by itself. By setting the two delineation equals, we can see that they intersect at \(x = 2\) So that the inequalities that define the range \(D\) in the range \(xy\) plane, is, \[\begin{array}{c} - 2 \le x \le 2\\ {x^2} \le y \le 8 - {x^2}\end{array}\] The volume is then set by ,\[\begin{*align}V &= \iint\limits_{D}{16xy + 200\ ,dA}\\ & = \int_{{ - 2}}^{{,2}}{\int_{{x^2}}}^{8 - {x^2}}{{16xy + 200\,dy}}\,dx}}\\ & = \int_{{ - 2}^{{\ ,2}}{\left. {\left({8x{y^2} + 200y} \right)} \right|_{{x^2}}^8 - {x^2}\,dx}}\\ & = \int_{{ { - 2}}^{{,2}}{{ - 128{x^3} - 400{x^2} + 512x + 1600\,dx}}\\ & = \left. {\left({ - 32{x^4} - \frac{400}}{3}{x^3} + 256{x^2} + 1600x} \right)} \right|_{ - 2}^2 = \frac{{12800}}{3}\end{align*}Preview] 4 Find volume for fixed, fixed, surrounded by planes \(4x + 2y + z = 10\), \(y = 3x\), \(z = 0\), \(x = 0\). View solution This example is slightly different from the previous one. Here the area \(D\) is not explicitly specified, so we have to find it. First, note that the last two planes really tell that we will not walk past and the plane \(yz\)-when we reach them. The first level, \(4x + 2y + z = 10\), is the top of the volume, and then we are really looking for volume under, \[z = 10 - 4x - 2y\] and over the region \(D\) in the \(xy\) aircraft. The second plane, \(y = 3x\) (yes, it's a plane), provides one of the sides of the volume as shown below. The area \(D\) is the area in the area \(xy\) plane (i.e. \(z = 0\)), bounded by \(y = 3x\), \(x = 0\) and the line where \(z + 4x + 2y = 10\) cuts the \(xy\) plane. We can determine where \ (z + 4x + 2y = 10\) cuts \(xy\) plane by connecting \(z = 0\) to it. \[0 + 4x + 2y = 10\hspace{0.25in} \Rightarrow \hspace{0.25in}2x + y = 5\hspace{0.25in} \Rightarrow \hspace{0.25in}y= - 2x + 5\] So here's a sketch region \(D\). The area \(D\) is really where this fixed will sit at \(xy\) level and here are the inequalities that define the region. \[\begin{array}{c}0 \le x \le 1\\ 3x \le y \le - 2x + 5\end{array}\] Here is the volume for this solid problem. \[\begin{align*}V & = \iint\limits_{D}{{10 - 4x - 2y\,dA}}\\ & = \int_{0}^{{\,1}}{\int_{{3x}}^{{ - 2x + 5}}{{10 - 4x - 2y\,dy}}\,dx}}\\ & \int_{0}^{{\,1}}{{{left. {\left({10y - 4xy - {y^2}} \right)} \right|_{3x}^{ - 2x + 5}\,dx}}\\ & = \int_{0}^{{,1}}{{25{x^2} - 50x + 25\,dx}}\\\ & | {\left({\frac{{25}}{3}{x^3} - 25{x^2} + 25x} \right)} \right|_0^1 = \frac{{25}}{3}\end{align*}\] Note, that more generally, \[V = \iint\limits_{D}{{{f\left({x,y} \right)\,dA}}\] provides the net volume between the graph for \(z = f\left({x,y} \right)\) and the range \(D\) in plane \(xy\) plane. Areas below the \(xy\) plane plan have a negative volume, and areas above planet \(xy\) planet have a positive volume. We saw a similar idea in Calculus I, where \[A = \int_{a}^{b}{{f\left(x \right)\,dx}}\] provides the net range between the path specified by \(y = f\left(\right)\) and the \(x\) axis in the range \(\left[{a,b} \right]\). The second geometric interpretation of a double integral is the following: \[{\mbox{Area of }}D = \iint\limits_{D}{dA}}\] This is easy to see why this is generally the case. Let's assume we want to find the area of the region shown below. From Calculus I know that this area can be found by the integral, \[A = \int_{{\,a}}^{{,b}}{{g_2}\left(x \right) - {g_1}\left(\right)\,dx}}\] Or in the form of a double, integral, we have, integral \[\begin*}{\mbox{Area of }}D & = \iint\limits_{D}{{dA}}\\\ & \ int_{{{\,a}}^{{{\{\{b}{{int_{{g_1{g_1}\left(left)}{{{\\,{g_2 left(x \right)}}{{dy}}\ .dx}}\\ & = \int_{{\ ,a}}^{{,b}} {{\left. y \right|_{g_1}\left(x \right)}^{g_2}\left(\højre)}\,dx}} = \int_{{{\,a}}^{{\,b}}{{{g_2}\left(x \right) - {g_1}\left(x \right)\,dx}}\end{align*}\] This is exactly the same formula, as we had in Calculus I. I.