Rudin real and complex analysis solutions chapter 10

1. The following fact has been tacitly used in this chapter: If $$A and $$B are separated subsets of the plane, if $$A is compact, and if $$B is closed, then there is $\delta > 0$ so much that $|\alpha - \beta | \geq \delta$ for all $\alpha\ in A$ and $\beta \ in B $. Prove this with any metric space in place of the aircraft. Proof. Let A is a compact kit and B is closed in a metric zone, so $A\cap = \varnothing$. Let $\delta = \inf_{\alpha,\beta}d(+alpha,\beta)$ where the movie is accepted with all $\alpha \in A$ and $\beta\in B$, and let $\{a_n\}$, $\b_n$} be two sequences at $$A and $$B respectively so that $\lim_{n\infty}d(a_n,b_n) = \delta$. Suppose $\delta = $0. Because $$A is compact, there is $c\in A$ and a subset $\a_{n_k}}}$$ $\{a_n\\}}$ so $\lim_{k\to\infty}d(a_{n_k}},c) = 0$. Therefore a_{n_k{n_k},c) = $\lim_ lim_{k\to\inft b_y}d(a_{n_k},b_{n_k}) = 0$. Therefore$c \in \bar{B}=B$, which is contrary to the hypothesis that $A\cap B = \varnothing$. So $\Delta > $0. We'il draw the conclusion. $\Box$ 2. Suppose $$f is a whole feature and that in each power series $$ f(z) = \sum_{n=0}^{\infty} c_n(z-a)n$$ at least one factor is 0. Show that $$f is polio. Hint. $n!c_n = f^ {(n)}(a)$. Proof. Let $a\in \mathbb{C}$. Consider the series from $$f to $$a, we have $$ f(z) = \sum\limits_{n=0}^{\infty}c_n(z-a)^n. $$By admission, $c_n = $0 for some $$n\leq 0. By $n!c_n = f^{(n)}(a)$, we get $f^{(n)}(a) = 0$ for some $$n\leq 0. Insert $K = \{z:f^ (n)}(z) = 0 \text{ for some }n\leq 0\}$, we have $a \in K$. This applies to any $a\in \mathbb{C}$. So the $K = \mathbb{C}$. Additionally, we have $$ K = \bigcup\limits_{n=0}^{\infty}\{z:f^ {(n)}(z) = 0\}. $$ If for every $$n, we don't have $f^ {(n)}(z) = $0 for every $z\in \mathbb{C}$, then $\{z:f^(n)}(z) = 0\}$ is a graphable set for all n. Therefore$K$ is a graphable set, which is a contradiction $K = \mathbb{C}$. So $f^ {(N)} = 0$ for some $N\geq 0$. Consider the $$f electric series at 0, we have $$f(z) = \limits_{n=0}^ {\infty}b_n z n^. $$ C $f^(n)}(0)=0$ for all $n\geq N$, we have $b_n = f^(n)}(n)}}}!!! = 0$ for all $n\geq N$. We'il draw the conclusion. $\Box$ 3. Suppose the $$f and $$g are entire features, and $|f(z)| \leq |g(z)|$ for each z. What conclusion can you draw? Proof. $f = CG$, where $| C|C|100 1 BGN If g1 does not disappear in $\mathbb{C}$, then $Z(g)$ there is no limit. Place $h = f/g$, then $h \in \mathbb{C}\backlash Z(g)$ and $$h is a meromorphic feature in $\mathbb{C}$. Let $a\ in Z(g)$. At the suggestion, $|h|\leq $1 in a deleted disk $D'(a,r)$ for some $$r >0$ such that $$geq 0 in $D'(a, r)$. So h has a removable singularity in $a $100. This is true for all $a\in Z(g)$, hence $h\in H(-mathbb{C})$ and $|h|\leq$ in $\mathbb{C}$. According to the of Liuville, h is a constant C at $\mathbb{C}$. So we're $f = Cg$ Cg$ ясно е, че $| B|C|100 1 лв. За случая $g =0$ в $\mathbb{C}$, получаваме $f = 0$ в $\mathbb{C}$, следователно $f= Cg$ с $C = 0$. Обратно, ако $f=Cg$ и $| C|\leq $1, след това $|f|\leq | C|| g| \leq |g|$. $\Box$ 7. Ако $f\in H(\Omega)$, формулата на Cauchy за производните на f, $$ f ^{(n)}(z) = \frac{n!} {2\pi i}\int_{\Gamma}\frac{f(\zeta)}} {(\zeta) ^{n+1}\,d\zeta\qquad (n=1,2,3,\ldots) $$ е валиден при определени условия на z и Γ. Дайте тези и докажете формулата. Доказателство. Γ е цикъл в $\Омега$, $\mathrm{{{}=0$ за всички $aotin \Омега$, $z\в \Омега\backslash \Gamma$, и $\mathrm{{{}}(z) = 1$. Чрез формулата на Коши, при горните условия, имаме $$ f(w) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(zeta)}{zeta-w},d zeta\\qquad (w\in\%%%% backslash\Gamma). $$ Нека $r > 0$ така, че $D(z;r)\subset\Omega\backslashlash\Gamma$. За $w \в D'(z;r/2)$, имаме $$ $ $ $ $ frac {f(w)-f(z)}} = \frac{1}{2\pi i(w-z)}\int_{\Gamma}f(\zeta)\наляво(\frac{1}{\zeta-w} - frac{1}{\zeta-\zeta-\\ zeta-\ , d\zeta. Оставете $w за да се печем. $$ lim_ {w\to z}\frac{1}{w-z}\left(\frac{1}{zeta-w} - \frac{1}{\zeta - z^\right) = \наляво(\frac{1}{\zeta-z}-надясно)' = \frac{1}{(-z^2^1}. $$ Освен това, за всички $w \в D'(z;r/2)$, имаме $$ \ляво|\frac{f(\zeta)}{w-z}\left(\frac{1}{\zeta-w} - \frac{1}{\zeta - z}\right)\! = \наляво|\frac{ f(\zeta)}} {(\zeta-w) (\zeta-z)}\надясно|\leq \frac{4}{r^2} \|f \|| _=_=\Gamma}\qquad(\zeta\in\Gamma). $$ Лесен аргумент, чрез прилагане на доминираната теорема за сближаване показва, че $$ $ f'(z) = \lim_{w\to z}\f(f(f)-f(z)}{w-z} =\frac{1}{2\pi and}\int_\{Gamma}\frac{f(zeta)}{\\\-z)zeta 2}\,d\zeta. $$ Ние сме доказали необходимата формула за случая $n= $ 1 . Забележете, че $$ \lim_{w\to z}\frac{1}{w-z}\наляво(\frac{1}{zeta-w)^n} - \frac{1}{(Zeta-z)^n}\надясно) = \\frac наляво(\frac{1}{(zeta-z)^ n}\надясно)' = \frac{1}{(zeta-z) ^ {n +1}} $, а за всички $w \в D'(z;r/2)$, имаме $$ \begin{ подравнени: \\наляво|\frac{f(\zeta)}{ w-z}\наляво(\frac{1}{(zeta-w)^n} - \frac{1}{(zeta-z)^n}\надясно)\надясно | &= \ляво|\frac{ f(\zeta)\sum_{i=0}{n-1}(\zeta-w)^i(\zeta-z)^ {зета-1}}{(zeta-w)^n(zeta\-n}}n}}n}}дясно| \\ &\ > наляво |f(\zeta)\limits_{i=0}^ {n-1}\frac{1}{(zeta-w)^{n-i}(\zeta-z) ^{ and}}\дясна|\\ \\\2^n}{ r^n}\|f\|==Gamma}\qquad (\zeta\in\Gamma). \end{aligned} $$ $$ По индукционна процедура, горните бележки и подобен аргумент, какъвто е случаят $n = $1, можем да докажем необходимата формула за произволни $n \geq $1. $\Box $19. Да предположим, че $f\ в H(U)$, $g\ в H(U)$, и нито f или g има нула в U. Ако $$ \frac{ f}{ f}\left(\frac{1}{n}\надясно) = \frac{ g}{g}{g}{g}{наляво(\frac{1}{n}\right)\qqua d (n=2,3,\ldots), $$ намери друга проста връзка между f и g. Доказателство. $f=Cg$, където $Ceq $0. Поставете $h = f/g$. Ясно е, че $h\в H(U)$. Освен това, с $$ h'(z) = \frac{f'(z)g(z)- f(z)g'(z)}{g^2(z)}, $$ ние $h{1}$${1} n{1}${1}$$$0{1}$$$$3{1} as $n \ for $ infty$, we get $h' (z) = 0 $ for every $z \ in U$. So $h(z) = h(0) + int_{[0,z]}}h'(w)\,dw = h(0)$ for all $z \in \mathbb{C}$. Put $C = h(0) =f(0)/g(0)0$, we get $f = CG$. If $f = Cg$ and $Ceq 0$, we get $f'(z)/f(z) = g'(z)/g(z)$for all $z\in U$, in particular for $\frac{1}{n}$, $n=2,3,\ldots$\$\$21. Assume that the $f\in H(\Omega)$, Ω contains the closed packing element and $|f(z)| < $1 if it is $|z|=1$. How many fixed points should f in the disk? That is, how many solutions does the $f (z)=z$ have there? Proof. 1 solution. Let g and $$h be two functions in Ω, defined by $g(z) = f(z) - z$ and $h(z) = -z$ for every $z\in \Omega$. Allowed , $ |g(z) - h(z)| < |h(z)| $ for all z in the circle $\{z:|z|=1\}$. Through Rouché's theorem, we get the number of decisions of $$g in $D(0;1)$ equal to the number of decisions of $$h in $D(0;1)$, which contains only one solution $z = 0 $. We'il draw the conclusion. $\Box$ 1. Suppose the $$u and $$v are truly harmonious features in the Ωaircraft region. Under what conditions is $uv $harmonica? (Note that the answer depends heavily on the fact that the question is the question of real functions.) Show that $$u^2 can't be harmonious at Ωunless $u is constant. For which $f\in H(\Omega)$ is $|f|^2$ harmonica? Proof. u is constant or $$v is constant or there is some real constant $Ceq 0$, so $u_x = Cv_y$, $u_y = -Cv_x$ (in other words, $u +iCv$ is holomorphic). $f $100,000 is constant. Note that each harmonious function has continuous partial derivatives of all orders. Suppose the $$uv is a harmonica. which means $(uv)_{xx}+(uv)__{yy}=0$ in Ω, of which $u_xv_xv_x + u_yv_y = 0$ in Ω (because $u_{xx}+u_{yy}=0$ and $v_{xx} +v_{yy}0$). that more that $$u and $$v are not permanent iv_x v_y $h iu_y u_x $g $h $g. If $Z(h) = \Omega$, then $v_x = v_y = $0 $v in $\ Omega$. $$D is a real part of the $$V holomorphic feature at $$D. $$V = v_x - $iv_y, we have $$V= $$D, therefore $$V is constant at $$D (for every z , w at $$D, we have $V(z)-V(w) = \int_{[z,w]}V'(\zeta)\,d\=0$), hence $v = \mathrm{const}$ in D. This argument shows that for every $z in Omega$, $$v is constant in some neighborhoods of z. Select $z_0 in \Omega$. Now it's easy to see that the $\ {z set in \Omega: v(z) = v(z_0)}$is both closed and open in Ω, therefore it's exactly Ω from the Ωconnectivity. This means that the v is constant at Ω, contrary to our assumption. So there $Z(h)$ there should be a point limit in Ω. Put $k = g/h$, we have k is a meromorphic function in $\ Omega$. But $$ k = \frac{g}{h}=h}=u_x - iu_y}{v_y+iv_x} = - u_yv_x}{v_x^2 + v_y^2}\in mathbb{R}. \qquad (*) $$ Там не не all non-implicit open subset of $\mathbb{R}$. So the k is constant in any disk where there is $g zero (from the open card theorem). Now, fix $a\ in Z(h)$. Let $r>0$ so that $D(0;r)\sub-set \Omega$ and $h(z)eq 0$ for every $$z in $D' (0;r)$. Then we have $$k is constant on each of the four discs $D(a+r/2;r/2)$, $D(a+ir/2;r/2)$, $D(a-r/2;r/2)$, $D(a-ir/2;r/2)$. Since the first disc intersects with the second, the second crosses with the third, and the third crosses with the fourth, we conclude that $$k is permanent in the union of these four discs, therefore it is constant at $D(a;r/2)$, because $D(a;r/2)$is a subgroup of this alliance. So the a is interchangeable singularity and k is constant in the neighborhood $D(a;r/2)$ from $$a. Through the connectivity of Ω, we get k is constant at $\ Omega$. Mark this constant with C. With $(*),, $C\in \mathbb{R}$, $Ceq 0$ (if not, $u_x u_y=0$ in Ω, therefore $$u is constant) and $u_x = Cv_y$, $u_y = $Cv_x. We get the answer to the first question. Now, if $$u^2 is harmornic in Ω, then, as above, we have $$u is constant or $u_x = $Cu_y, $u_y= -Cu_x$ for some real constant $Ceq 0 $. The second case results in $u_x = u_y = 0$ in Ω, which also implies that u is constant. We confirm the second statement. (A simpler way to get this confirmation is that we take a laplase of u^2 and get $u_x = u_y =0$.) For the last question, $f = u + iv $. Since we $f H(\Omega)$, we have $$u and $$v are harmonic in $\ Omega$. Now $|f|^2$ is $u^ 2+ v^2$ is harmonious. Take laplacesian on $u^ ^ 2 + v ^ 2$, we get $u_x = v_x = u_y = v_y =0$. So the $u $ and $v $ are constant, therefore the $f $ is constant. $\Box$ 2. Suppose $$f is a complex feature in region Ω, and $$f and $$f^2$ are harmonious at Ω. Prove that either $$f or $\bar{f}$ is holomorphic at Ω. Proof. Place $f = u + iv$. Because $f $1 is harmonious, $u $$v$100 is harmonious. Suppose the $2 $f is harmonious at Ω, suggesting $$u^2 - v ^2$ and $$uv are harmonious. Through Exercise 1, $$uv, a harmonica leads to $$u is constant or $$v is a constant or there is a non-zero real constant C so that $u_x = Cv_y and $u_y = -$Cv_x. If u is constant, then $$v^2 is harmonious (because $$u^2-v^2$ is harmonious), therefore $$v is constant from Excercise 1. Similarly, $$v is constant, we conclude that $$f is permanent if $$u or $$v is permanent, therefore both $$f and $\bar{f}$ is holomorphic. Suppose the $$f is not a constant. Take a laplasia from u^2-v^2, we get $(u_x) ^ 2 +(u_y) = (v_x)^2 + (v_y)^2$. Replace $u_x = Cv_y$ and $u_y = -Cv_x$ to this equation, we get $C^2 = 1$ unless $v_x = v_y = 0$ in Ω. The $C = $1 leads to the $$f equations, and the f is holomorphic. The $C=-1$ results in the $\bar{f}$, therefore $\ holomorphic. We'il draw the conclusion. (There is another way that does not benefit from first question in Excercise 1. Take laplacesian from $u^2-v^2$2. and uv, we get $(u_x)^2+(u_y)^2= (v_x) ^ 2 + (v_y) ^ 2$ and $u_xv_x + u_yv_y =0$, which implies $(u_x+iv_x)^2 + (u_y +iv_y)^2 = 0$. Therefore, $$ (u_x+v_y+iv_x - i u_y)u_x -v_y+iv_x+iu_y) = 0. $$ Use $ab=0 \Leftarrow a\bar{b}=0$, we get $$ (u_x+v_y+iv_x - i u_y)u_x -v_y-iv_x-iu_y) = 0. $$ Put $g = u_x+v_y+iv_x - i u_y$ and $h = u_x -v_y-ivx-iu_y$, we get $gh= 0$ It's easy to check if $$g and $$h are holomorphic (by checking the Koshi-Riman equations, we get $$u_x-iu_y$ and $$v_y+iv_x$ are holomorphic). We claim that $g=0$ in Ω or $h = 0 $ in $\ Omega$. Suppose the opposite, because $Z(g)$ and $Z(h)$are counting sets, $Z(gh)$ should be a counting set, which is contrary to $Z(GH)=\Omega$. The $g=0$ leads to f is holomorphic. The $h=$0 leads to $\bar{f}$is holomorphic. $\Box$ 3. If u is a harmonious feature in a region Ω, how about a set of points where the $$u gradient is $0? (This is the set on which $u_x = u_y = $0.) Proof. $K = \{z:u_x(z) = u_y(z) = 0\}$ no limit in Ω (therefore there are most counting items) or $K = Omega$. Put $f = u_x - I u_y $. If u1 is the harmonium, f100 is a holomorphic function, checking out Kashi-Ryman's equations. Moreover, because $K = Z(f)$, we get the conclusion. (The $K = \Omega$ results in $u_x = u_y = $0 $u in $\ Omega$. $\Box$ 1. Suppose Δ is a closed equinox triangle on the plane, with a, $$b, $$c. Find $\max (|z-a|| z-b|| $1.4. as $z $1.4 billion range above Δ. Proof. $\max = \sqrt{3}|b-a|^3/8$. До максималната модулна теорем, имаме $$ \max\limits_{ z \ in\Delta}(|(z-a(z-b)(z-c)|) = \max\limits_{ z\in\част\Делта}(|(z-a(z-b)(z-c)|). $$ Сега е елементарно да се намери максимума на дясната страна, което се постига, когато $z $ е една от средните точки на ръбовете ab, bc, и ca. Наистина, без загуба на общото, да предположим, че максимумът е постигнат на $z $ на $ab $. Поставете $x = |z-a|$ и $l = |b-a|$$. Така че максималната е $$ x (l-x)\sqrt{(x-l/2) ^ 2 + (\sqrt{3}l/2) ^ 2= x(l-x)\sqrt{l^2-x(l-x)}}. $$ Когато x диапазони над $ [0,l] $, $x(l-x)$ диапазони над $ [0, l ^2/4]$. Поставете $t = x(l-x)$, ние получаваме максимума е $t\sqrt {l^2-t}$, или $\sqrt {l^ 2t^2t^2t^2t^3}$. Производната на $l^2t^2t^3$ според t е $(2l^2-3t$, which is greater than or equal to 0 when t ranges above $[0,l^2/4]$. So $t\sqrt{l^2-t}$ increases when $$t is run from 0 to $l^2/4$, and the maximum is achieved when $t = l ^2/4$, which is so, $x = l/2$. $\Box$ 3. Suppose you $f H(Omega)$. Under what conditions can $|f|$ have a local minimum in Ω. Proof. f is constant or f has at least one zero in Ω. Suppose the $$f is not permanent and there is no zero in Ω. So $f^-1}\in H(\Omega)$ and $|f|$ has a minimum in $$a$ iff $|f^{{-1}|$ there is a local local a. By applying the maximum modular theorem, if $|f^{-1}|$ has a local maximum, then $f^ {-1}$ must be permanent, therefore $ $f is constant. So $f doesn't have a local minimum of $\ Omega$. (For more details, we believe that the $f $$f case is $f not a constant $a. If $a 0, then $$feq 0 in an open district of $$a. As above, $$f should be permanent in this neighborhood, therefore be permanent in $\Omega $. So, in this case, we can see that the set of all local minimals of $$f is $$Z(f)$, a set of all zeros of $$f.) $\Box$ 4. (a) Assume that Ω is a region, $$D is a disk, $\bar {D}\subset \Omega$, $f\in H (\Omega)$, $$f is not a constant, and $|f|$ is constant at the $$D limit. Show that the $$f has at least one zero in the D. (b) Find all entire features f such that $|f(z)| = $1 each time $|z|= $1. Proof. (a)Suppose there is $f zero in the $$D. If $|f|= 0$ to ∂D, then from the maximum theorem module, $|f|=0$ in $$D, therefore f is constant. So you $feq $100 to $\part of D$, so $f eq 0$ to $\ bar{D}$. Insert $c>0$ is the constant value of $|f|$to ∂D. To the maximum modular theorem, we have $|f|\leq c$ in D. Consider the f^{-1} at $\bar{D}$. It is clear that $f^ {-1}\in C(\bar{D})$ and $f^{-1}\in H(D)$. Up to the maximum modular theorem, we have $|f ^{-1}| \leq 1/c$ in D, therefore $|f| \geq c$ in D. So $|f| = C$ in D, $f(D) \subset \partD(0;c)$. Through the open mapping theorem, f should be constant at $$D, therefore to be constant at Ω. This contradiction indicates that the $f should have at least one zero in the $D dollars. (In this evidence, we just need f is holomorphic at $$D$ and continuous at $\bar{D}$.) (b) f is from the cz^m, where $|c| = 1$ and $m \geq 0$. Suppose the $$f is not permanent. With (a), f has at least one zero in $D(0;1)$. The set of all zeros of $f has no limit in $\mathbb{C}$. Because $\bar{D}(0;1)$ is compact, the number of zeros in f in $D(0;1)$is extreme. Let $a_1, a_2,a_n\in D(0;1)$ be such zeros and $$m_1,m_2,\ldots,$m_n are zero orders of $$f at these points. Insert $$\begin{aligned} g(z) &= f(z)\prod\limits_{i=1}^ n \left(\1-bar{1-bar{a}_iz}{ z-a_i}\{ m_i}\\\\\\\\\\frac{f(z)}{(z-bar{a}_1)} m_1 m_i}\\\\\\\\frac{f(z)}{z-bar{a}_1)} m_i}\\\\\\\\frac{f(z)}{z-bar{a}_1)} m_i}\\\\\\\\frac{f(z)}{z-bar{a}_1)} m_2 (z-\bar _n{m_n}}}limits_{i= $z _iz 1}^ It is clear that $g\in H(\mathbb{C})$ and $$g have zero in $D(0;1)$. In addition, about $|z| = 1$, we have $$ \left|\frac{1-\bar{a}_iz}{z-a_i}\right| = \left|z.\frac{1/z-bar{a}_i}{ z-a_i} \right| = \left|z.\frac{\bar{z}-bar{a}_i}{ z-a_i} \right| = \left|z.\fracline{\overline{z - a_i}} {z-a_i} \right| = 1\qquad (i=\ overline{1, n}). $$ $|g(z)|||| =1$ for each $z\in \partial With (a), g must be a constant, which will be marked with c. Apparently $|c|=$1. If there is a i такива, че $a_ieq 0$, след това $g (1/\bar{a}_i) = 0 $, което е противоречие на $|c|=1$. Така $n=1$ и $a_1 = 0$. Това означава, че $f(z) = cz ^m_1}$, където $|c|=1$ и $m_1 > 0$. Заедно със случая f е постоянна, виждаме, че f трябва да бъде във формата cz^m}, където $|c|=1$ и $m_geq 0$. В крайна сметка е ясно, че тези функции отговарят на изискванията ни. $\Box$ 5. Да предположим, че Ω е ограничена област, $\ f_n\} $ е последователност от непрекъснати функции на $\bar{\Omega}$, които са холоморфни в $\Омега$, и $\{f_n\}$ се конверсира равномерно на границата от Ω. Докажете, че $\{f_n}$$ се свива равномерно на $\bar{\Omega}$. Доказателство. Оправям $\епсилон > 0$. Има N достатъчно голям, че за всеки $m,n > N$, имаме $f_n(z) - f_m(z)| < \epsilon$ за всеки $z\ част част \Омега$. От максималната теорем на модула, за всеки $m, n>N$, имаме $ $ f_n(z) - f_m(z)| < \epsilon \qquad (*) $$ for every $z\in \bar{\Omega}$. So for each $z\in \partial\Omega$, $\{f_n(z)\}$ is a Cauchy sequence, hence converges. For each z, put $f(z)$ be this limit. Let m converges to ∞ in $(*)$, we get the conclusion that $\{f_n\}$ converges to f uniformly on $\bar{\Omega}$. $\Box$ 6. Suppose $f\in H(\Omega)$, Γ is a cycle in Ω such that $\mathrm{Ind}_{\Gamma}(\alpha) = 0$ for all $\alpha otin \Omega$, $|f(\zeta)|\leq 1$ for every $\zeta \in\Gamma^*$, and $\mathrm{Ind}_{\Gamma}(z) eq 0$. Prove that $|f(z)|\leq 1$. Proof. Let U be an open connected component of $\mathbb{C}\backslash \Gamma$ such that the index of any point in U with respect to Γ is not zero. So U is bounded and, by assumption, $U \subset \Omega$. Let $z \in \partial U$. Because $z otin U$ and also z can not be in any other component of $\mathbb{C}\backslash \Gamma$, we have $z\in \Gamma$. So $\partial U\subset \Gamma$ and $\bar{U} \subset \Omega$. By assumption, we get $\|f\|_{\partial U} \leq 1$. Moreover, because U is bounded, we have $\bar{U}$ is compact. By the maximum modulus theorem, we get $|f(z)| \leq 1$ for every $z\in U$. So $|f(z)|\leq 1$ for every z such that $\mathrm{Ind}(z) eq 0$. $\Box$ 7. In the proof of Theorem 12.8 it was tacitly assumed that $M(a)>0$ and $M(b) > 0$. Покажете, че теоремът е верен, ако $M(a) = 0 $, и че след това $f(z) = 0$ за всички $z\в \Омега$. Доказателство. Когато $M(a) = 0 $, помислете за сегмента $L = \{a+ yi: y\ in (0,1)\}$, ясно е, че f изчезват на L. Поставете $\Theta = \ Омега \ чаша L \cup \{ x+ yi: 2a - b < x < b\}$. По принцип на отражението schwarz получаваме функция $F \in H(\Theta)$, така че $F = f$ на $\ Omega \ cup L$. Тъй като Θ е регион и F изчезва на L, $F $ трябва да се изчезват Θ, следователно f изчезват на $\Омега$. С други думи, имаме $M(x) = 0$ за всички $x \ в (a, b)$. (За да потвърдим, че можем да приложим принципа на размисъл тук, помислете за картата $z \to i(z-a)$.) $\Box $ 15. Да предположим, $f\in $f\in Prove that there is a sequence of $\{z_n\}$ in $$U, such as that $|z_n|\to $1 and $\{f(z_n)}$$ is limited. Proof. The problem is clear about the case $f $1 is constant. For the case$f there's no $$U in the case. For each $r\c (0,1) $, Apply the maximum modular theorem for the function f^{-1} with domain $\bar{D}(0;r)$, we get $|f(0)|min_{z\in\part D(0;r)}|f(z)$, so we get a point a_r, so $|a_r| = r$ and $|f(a_r)|\leq |f(0)|$. Thus, the sequence $\{a_{1-1/n}\}$ meets our requirements. Now let's assume that $f $1 is not permanent and $f $100 has at least one zero in $U $. Place $Z(s)$ is a set of zeros of $$f in U. It is clear that $Z(s)$has no point limit in U. If $Z(f)$ is an unlimited set, then it must have a limit point in $\bar{U}$, which is actually $\part U$. So we get sequence $\{{z_n}\subset U$ conversion to this limit point (hence $z_n|\to 1$) and $|f(z_n)|=0$ for every $$n. We will conclude this case. The remaining case is that $Z(e)$ is the final set. Insert $Z(f) = \{a_1,a_2,\ldots,a_n\}$ and let $m_1,m_2,\ldots,m_n$ are zero orders of $$f on $a_1, a_2, \ldots, a_n$, respectively. Insert $$g(z) = \frac{f(z)}} {(z-a_1)^{m_1}(z-a_2)^{m_2}\ldots(z-a_n)}^m_n}}}}\qquad (z\in U). $$ It is clear that $g\in H(U)$ and g there is no zero in U. As above, we get a sequence of $\{z_n\}$ in $$U, such as $$z_n|\to $1 and $|g(z_n)| < M$ for $M > 0$. About $|z|| >\max\{{a_1|| a_2|,\ldots,|a_n|}} $, we have $$\begin{ aligned} |f(z)| "= |g(z)|| z_n-a_1|^^ ^^ m_1|z_n-a_2| {m_2}\ldots|z-a_n|^{m_n}\ (1-|a_1|) {m_1} (1-|a_2|) {m_2}\ldo(1-|a_n|) ^{m_n}. \end{aligned} $$ So for $$n big enough, we have $|z_n| >\max\{|a_1|,|a_2|,\a_n|}} and $$ |f(z_n)| < M(1-|a_1|) ^m_1}(1-|a_2|) ^{ m_2}\Ldots(1-|a_n|) ^m_n}. $$ The sequence $\{{z_n}$is the one we want to find. $\Box$16. Suppose Ω is a limited region, $f\in H(\Omega)$, and $$ \limsup\limits_{n\to\infty} |f(z_n)| \leq M $$ for each series $\{z_n\}$in Ω, which shrinks to a limit point of Ω. that $|0 f(z)|\leq M$ for all $z\in \%> $z% Instead, adjust $\epsilon >0$, we will prove that the set $A = \{z\in \Omega: |f(z)| > M+\epsilon\}}$ is empty. On the contrary, suppose the $A $ is not empty. It is clear that $A $ is open. Let U be an inappropriate connected open component of $$A and let's $z\in partial U\subset\bar{\Omega}$. There is a series $\{z_n\}\subset U$ such that $z_n \to z and $|f(z_n)|> M+\epsilon$ for all n. Then we have $\limsup_{n\to} |f(z_n)| \geq M + \epsilon$, which is contrary to our hypothesis if $z\in partial \Omega$. So you $z \Omega$ for all $z in partial U$. In other words, $\bar{U}\subset \Omega$. Since Ω is limited, we have $\bar{U}$ is a compact kit. Moreover, for $z\ in partial U$, we \geq M + \epsilon$, hence $|f(z)| = M +\epsilon$ (if not then $z\in A$, therefore $z\ in A\backslash U$, which is an open set, hence $zotin \{U}$). Apply the maximum modular theorem for $$f with domain $\bar {U}$, we get $|f(z)|\leq M + \epsilon$ for every $z\in U$, which is contrary to our definition of $$U. So A100 should be empty. $\Box$ 1. Prove that any meromorphic function of S^2 is rational. Proof. Note that $$f must have an isolated singularity at ∞, which means that $$f is holomorphic in $D'(\infty;r)=\{z\in\mathbb{C}:|z|> r\}$. This means that the set of all $$f poles in $\mathbb{C}$is final (because this set has no limit and is a subset of the $\overline{D}(0;r)$) compact set). If we subtract from $f $1 the sum of all the main parts of these pillars, we get an entire function $g $ with isolated exclusivity at $\ infty $. Of course, this singularity of $$g∞ cannot be an essential singularity due to our hypothesis that $$f is meromorphic at $$S^2. If this singularity is interchangeable singularity, then $$g is limited in a $$infty neighborhood, therefore it's limited to $\mathbb{C}$, so it's constant from Liouville's theorem. If this singularity is a pole, the bulk of $g$$infty is polinomyal. By subtracting this polynom g100, we again get an entire feature with a mobile singularity of $\ infty $. Both cases lead to the same conclusion that $$f is a rational function. $\Box$ 2. Let $\Omega = \{z:|z|<1 \text{= and= }|2z-1|=> 1\}$, and I guess $f\in H(Omega)$. (a) There must be such a sequence of P_n polinomas, so that $P_n\f$ equal to compact subs of Ω? (b) Should there be such a sequence that converges $f $100 000 in Ω? (c)Changed the answer to (b) if more than $$f is required, namely that $$f to be holomorphic in an open set that contains the closing of Ω? Proof. (a)Yes. Because the $S^2\backslash \Omega$ is connected (then apply a Runge theorem). (b) No. See. (c) or simply, consider the function $f(z) = z ^{{-1}$ with a similar argument. (c) No. Consider the function $f(z) = (z--fuff{1}{10})^ - 1}$. Suppose there is a sequence of $P_n P_n so that P_n converges to f equally in Ω. That is, there is a $P $100,000 so $$ | P(z) - (z-1/10) ^-1}| < 1= $$= for= every= $z\in= \omega$.= fix= $z= \in= \partial= d(0;1)$.= let= $\{z_n\}$= be= a= sequence= in= ω= converges= to= z.= we= have= $$\begin{aligned}= |p(z)|=\lim\limits_{n\to\infty}| P(z_n)| &\leq= 1= += \lim\limits_{n\to\infty}\frac{1}{\left|= z_n-1/10\right|= }=1 += \frac{1}{\left|= z-1/10\right|= }\\= &\leq= 1= += \frac{1}{1= -= 1/10}=>< 3.= \end{aligned}= $$= by= the= modulus= theorem,= $|p(z)|=>< 3$= for= every= $z= \in= d(0;1)$.= however,= for= $z=\frac{-1}{10}$, we= have= $$= |p(-1/10)|=> \frac{1}{\left|-1/10 - 1/10\дясно|} - 1 = 4. $$ Това </1> </1> $(z---frac{1}{10}) $1-$$1 Box3. $$P, $1.00, $P $1,$n -1.2.3,ldots, $P $0, $0, $0, $zeq $0, $0, $0, $$n $? Доказателство. Да. $$n, $$K$$D-n-cup L_n'cup-cup-$, '$D'n' 'bar'd(0;n): 'mathrm'Im'z-geq 'frac{1}'n'$, $L'n 'x'yi: Y-0, 'frac{1}'n'leq 'x'leq n.'nq$.... $$U$$D-L_n $$V$0, $0, $0, $0, $0, $0, $0, $0, $0, $$U $$V$. $0 to $ $f V_n U_n 0 to $0, $ $U 0, $1,$$1 to $1 V. '$S '2' 'K_n' '$', 'K_n' Runge, $$P, $07, $07, $1, $1, $1, $1, P_n (z) - f_n (z) 'Frac{1}'n$' $z $K_n, $15 P_n (z) 'Frac{1}'n$' $z D_n $L_n $00.00 P_n (0)-1 Frac{1}$. It's not going to be a problem. $D_n L_n , $-math-backslash, $-c-backslash, $-c-backslash, $-C-backslash, '$P'n' -$P'n(to 1) 'to 1 $$n $$z eq $0, $$P$0 - $0- $0- $$n $0. $$P -$P-P_n(0) - $1, $1, Box4. $$P $$$$$$$$$1.limits_ $. P_n (z) 'begin'cases' 1 'text' It's not going to be the one. 2000000 000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000-0000000s 0 $0 $$. Да. $$n, $$K$$D-cup E_n-cup L_n, $D -n-z-in-bar-D(0;n): '{1}' $E 'n' 'z-in-bar'D(0;n):'Mathrm'Im'n'leq-frac{1} '$L'n' 'x' yi:y'0, 'x's 'leq n'$'. $$U$$D,$V$$E$$$W$$L$, $$U,V, $$W$. '$f'n' in H (U_n-cup V_n 'cup W_n') $1 -$1,U, $1 -$$V$, $0-$$L.s. '$S '2' 'K_n' '$', 'K_n' Runge, $$P, $07, $07, $1, $1, $1, $1, P_n(z)-f_n (z) 'Frac{1}'n$' $z $K_n, $15 P_n (z)-1 $$z$$z, $D_n {1} P_n (z) 'Frac{1}$$z $E_n, $19 P_n (z) It's ${1}$$z $L_n. $$D_n, $$E_n, $L_n,$, $1,00, $1,000, $1,000, $1,000, O't, '$$P's $1.' $Box $7. that in Theorem 13.9 we should not assume that A crosses each component of $S^2\backslash\Omega$. Singly enough to assume that closing the A crosses each component of $S^ 2\backslash \Omega$. Proof. We have Ω is an alliance from the series K_n is located inside $K_{n+1$$ for every $$n, each compact subset Ω is located in some K_n, and for every $$n, each component of $S^2+ backslash K_n$ contains a component of $S^2 \backslash \Omega$ (Theorem 13.3). Fix $n $. Follow the proof of theorem 13.9, it is enough to show that each component of the $S^2\back slash K_n contains a point of $$A. let U be a component of $S^2\backslash K_n U. It is clear that $$U is open $S. $U \cap Aq \varnothing$. $\Box$ 8. In which Ω is the entire plane, with a direct argument that does not appeal runge's theorem. Proof. For each n, simply denote $\bar{D}(0;n)$ K_n. Place $A_1 = A\cap K_1$, and for each $n \geq 2$, insert $A_n = A\cap(K_n\backslash K_{n-1})$. It's easy to see that every A_n is extreme. Now for every n, put $$Q_n(z) = \sum\limits_{\alpha\in A_n} P_{\alpha}(z). $$ For every n, we have Q_n is a rational function and the poles of Q_n lie in A_n. Fix $n \geq 2 $. We Q_n is holomorphic in an open set containing K_{n-1}. Consider presenting the series from $$Q to 0 and recall that K_{n-1} is $\bar{D}(0;n-1)$, we get the energy series to converge evenly in Q_n at K_{n-1}. That is, we have a R_n so much that $$ | R_n(z) - Q_n(z)| < 2^ ^{-n}\qquad (z\in K_{n-1}). $$ Follow the proof of Mitag-Leffler's theorem, we can see that the function $$f(z) = Q_1(z) + \sum_{n=2}^{infty}(Q_n(z) - R_n(z)\qquad (z\in \in\c}) $1000 has the desired properties. $\Box $ 9. Suppose Ω is just a connected region, $f\in H(\Omega)$, $$f has zero in Ω, and $$n is a positive integer. Prove that there is $g\in H(\Omega)$, so $g^n = f$. Proof. Let me h is a holomorphic logarithm of $$f in Ω (existence comes from the hypothesis that Ω is just a connected region, $f\in H(\Omega)$, and $$f has zero in Ω). Now the $g = \exp(h/n) $ is in $H(\Omega)$ and $g^ n = \exp(h) = f$. $\Box $ 10. Assume that Ω is a region $f\in H(\Omega)$, and $feq 0. Prove that $$f has a logarithm in Ω if and only if $$f has holomorphic n -th roots in Ω for each positive integer n. Proof. The $(Rightarrow)$ side is clear, $$n-a-root of $$f is $\exp(g/n)$, where $$g is a holomorphic logarithm at Ω f. For $(\Leftarrow) $ country, at the beginning, we will show that f does not have zero in $\ Omega$. Suppose you $f(a) = $0 for some $a\ Omega$. Since $$feq 0, we have $$f(z) = (z-a) ^m(z)$, where $$m>0 is in the order of zero $$f at $$a, $g\in H(Omega)$, and $g(a) eq 0$. Consider holomorphic $(m+1)$th root $varphi$ from $$f. Since $ \ varphi (a) = $ and $\ varphi eq 0 $, we have $\varphi(z) = = where $h\in H(\Omega)$. Therefore$g(z) = (z-a)h(z)^{m+1}$ and this gives us $g(a) = 0$ (contradiction). To show that $$f has a logarithm in 0%Omega$, it is enough to show that $$\int_{\gam}\frac{f'(z)}}\,dz = 0/qquad (*) $$$ for each closed path γ in Ω. Suppose $$$$ is true for every closed road in Ω. Adjust $z_0\in \Omega$ and let $$l_0 be logarithm at $$f(z_0)$ (this logarithm exists because $f(z_0)eq 0$). Let L is a feature in Ω, defined by $$L(z) = l_0 + int_ {\Gamma(z)}\frac{f'(zeta)}{f(zeta)}\d\zeta\qquad (z\in\Omega), $$ $ $ $ $ $ where $\Gamma(z)$is each time from $\$ from z_0 to z. Due to $/$, this feature is well defined. Fix $a\in \Omega$. For all $D(a;r)\subset\Omega$ and all $z\in D'(a;r)$, we have $$\frac{L(z)-L(a)}{z-a}-a}-f'(a)}}}{f(f))} = \frac{1}{z-a}\int_{[a,z]}\left(\frac{f'(zeta)}{ f(\zeta)}{f'(zeta)}}\frac{f'(a)}{f(a)}\right)\, I'm not going to be $$ Because f'/f is continuous at $a $, it is easy to see that the $L'(a) = f'a/f(a)$. So the $L\in H (\Omega)$ and the formula $L'=f'/f$gives us $(exp(L) / f)'= 0$. Therefore, $\exp(L)/f = \mathrm{const} = \exp(L(z_0))/f(z_0)= 1$. So L is a holomorphic logarithm $f $100 million in Ω. In the end, to show that $$$$ is true, we notice that $\frac{1}{2\pi i} $ times on the left side of $$$$$$ is the $f\circ$$$at 0, which is an integer. Освен това, ако g е холоморфен n-ти корен от f в $\Омега$, след това $$ \$ \frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}\,dz = \frac{n}{2\pi} \int_{\gamma}\frac{g'(z)}{g(z)\,dz. $$ Дясната страна е цяло число, което се дели на n. Ние получаваме заключението, като отбелязваме, че 0 е уникалното цяло число, което е делимо от всички положителни числа. $\Box $ 11. Да предположим, че $f_n\in H(\Omega)$ ($n= 1,2,3,\ldots$), f е сложна функция в $\ Омега$, и $f(z) = \lim_{n\to\infty}f_n(z)$за всеки $z\in Омега$. Докажи, че Ω има плътно отворено подмножество V, на което f е холоморфна. Намек. Поставете $\varphi = \sup|f_n|$$ Използвайте теоремите на Baire to prove that each disk in Ω contains a disk that has φlimited. Apply Drained 10.5. (In general, $Veq\Omega$. Compare Excercises 3 and 4.) Proof. Insert $\varphi(z) = \sup_n |f_n(z)|$ for each $z \in \Omega$. It is well defined because $\lim_{n\to\infty} f_n(z)$exists (and equals $f(z)$) for every $z \in Omega$. Fix B be closed disk at Ω (so $$B is finished). For each n, place $A_n = \{z\ in B:\varphi(z) \leq n\}$. From the definition of φ, it is clear that $$$ A_n = \bigcap\limits_{m=1}^ {\infty} \{{z\z\c: |f_m(z)|\leq n\}. $$ Therefore, $$A_n$ is a closed set at $$B for every $$n. There is $$n through Baire theorem, so $$A_n$ is nowhere near a dense subset of $$B. So A_n contains an open disk $U_B\subset B$ in which φ is from the above from n. Now, apply Exercise 10.5, we get $f \ in H (U_B) $. Let $$ $$ = \bigcup\limits_B U_B, $$ where $B $ ranges above all closed disks in Ω. It is clear that V is a dense open subset of Ω and $f\ in H(V)$. $\Box$ 10. Suppose $$f and $$g are holomorphic cards of $$U in $+, $$f is one to one, $f(U) = \Omega$, and $f(0) = g(0)$. Prove<$$ g(D(0;r)\sub-2d(0;0;r)\qquad (0< r < 1). Insert $h=f^ - 1}\circ g$. Then $h(U)\subsetS 1 and $h(0) = $0. By Schwartz lemma, we have $|h(z)|\leq |z|$ for all $z\in U$. Fix $r\ in (0,1)$. If $z\in D(0;r)$ it also $h(z)\in D(0;r)$, therefore $f^{-1}\circ g (z) \in D(0;r)$, and therefore $g(z) \in f(D;0;r))$. So $g (D(0;r)\f(D(0;r)$. $\Box$ 12. Suppose Ω is a convex area, $f\in H(\Omega)$, and $\mathrm{Re}f'(z)>0$ for all $z\in Omega$$f. (Turn off the trivial $f=$ constant.) Show by example that convex cannot be replaced with a simple connected one. Proof. $\Omega = \mathbb{C}\backslash [0,\infty)$, $f(z) = -iz^ {3/2}+z/sqrt{2}$. Let $a, get $a $1,000. We have $$ f(b) - f(a) = \int_{[a,b]}f'(z)\,dz = (b-a)\int_0^1 f'(a+(b-a)t)\, dt. $$ and then $$\begin{ unification} \mathrm{Re}\left[\frac{ f(b)-f(a)}{b-a}\right] &= \mathrm{Re}\left[\int_0^1 f'(a+(b-a)t)\, dt\right]\\\\\\\\\\\int_0^int_0^1 \mathrm{Re}\left[f'(a+(b-a)t)\right]\, dt > 0. \end{aligned} $$ So we need to have $f(b) eq f(a)$, and that means $$f is one-to-one. Now I guess $\mathrm{ Re}f'(z)\geq 0$ for all $z\in \Omega$. Let $Z = \{ z\ in \Omega: \mathrm{Re}f'(z) = 0\} $. Of course, $Z $ is a closed set. Under the average property $\mathrm{Re}f'$ and $\mathrm{Re}f'(z)/geq 0$ for all $$z\in \Omega$, it is easy to show that $$Z$ is an open set. So $Z = \ varnothing $ or $Z = \ Omega $. The first case suggests, as we f is one in one in Ω. The second case involves $f'(\Omega) \subset_in}____ through the open mapping theorization, $f' is constant at Ω, f is constant or linear. The linear case also leads to f1 to one. So unless it's a constant $$f f is always one-to-one when $\mathrm{Re}f'(z)\geq 0$ for all $$z\in \Omega$. For example, a stretch cannot be replaced with simply connected, consider $\Omega = \mathbb{C}\backslash [0,\infty)$ and $f(z) = -iz ^ {3/2}+z/\sqt{2}$, as the branch is cut for $z ^{1/2}$, is $[0,\infty$.$. It is easy to check if $f' (z) = -\frac{3i}{2}zz^1/2} + 1/\sqrt{2}$ has a positive real part and $f(i) = f (-i) $. $\Box $ 13. Suppose Ω is a region, $f_n\in H(\Omega)$ for $n= 1,2,3,\ldots$, each f_n is one-to-one in Ω, and $f_n\to f$ evenly on compact subsets of Prove that the $f is either a constant or one-to-one in Ω. Show that both cases can occur. Proof. Assume that there is $a,b\ in\Omega$, so $aeq b$ and $f(a) = f(b)$. Choose $r > 0$ 0$ \Омега$ и $b оттенят D(a;r)$. Според предположението имаме $f_n(z) - f_n(b) \да f(z) -f(b)$, равномерно на $\bar{D}(a;r)$. Сега от Упражнение 10.20 с бележка, че $f_n(z) - f_n(b) 0$ за всички $z\ в D(a;r)$ и всички n, имаме $f(z) - f(b)eq 0$ за всички $z\ в D(a;r)$ или $f (z) - f(b)=0$ за всички $z\ в D(a;r)$. Първият случай не може да се случи, защото $f(a) = f(b)$. И вторият случай предполага, че $f $ е постоянна. $\Box $ 16. Нека $\mathcal{F}$ да бъде клас на всички $f\in H(U)$, за които $$ \iint\limits_U |^2 \, d x\, d y \leq 1. Това нормално семейство ли е? Доказателство. Нека $a \в U$ и $R > 0$ такива, че $D(a;2R) \subset U$. За $z\in D(a; R)$ $, имаме $D(z;r)\subset U$ за всички $r\in (0, R]$ $. Под средното свойство на стойността на f, имаме $$ f(z) = \frac{1}{2\pi}\int_{0}^ {2\pi} f(z + re^{it}),dt.$$ Умножете двете страни с r и ги интегрирайте според t от 0 до R, получаваме $$ \frac{R^2}{2}f(z) = \frac{1}{2\pi}\int_0^R \int_0^int_0^{2\pi rf(z +^re^{it}\},dt.$$ Следователно, от неравенството на Hölder, ние имаме $$ \begin {подравнени} |f(z)| &\leq\frac{1}{\pi R^2}\наляво[\int_0^ int_0^ int_0^ {2\pi} r\, dr\, dt\надясно]^{1/2}\наляво]\int_0^int_0^1 22%, 1000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000{D; limits_{1}11000000000 R)}}|f(w)|\,dw\right]\ ^{1/2} \n\fc\frac{1} sqrt{\pi} R}. \end{ aligned} $$ за всеки компактен набор $K \, подмножество \Omega$, можем да го покрием с $D(a) R)$ с имота, който $D(a;2R) \подмятн U$. По горния аргумент $\mathcal{F}$ е равномерно ограничена във всеки диск, следователно е равномерно ограничена на K. Според Теорем 14.6, ние заключаваме, че $\mathcal{F}$ е нормално семейство. $\Box$$\Box$