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Gradient divergence and curl solved problems

In this final section we will establish some relationships between the gradient, divergence and bunch, and also introduce a new amount called Laplacian. Then we will show you how to write these quantities in cylindrical and spherical coordinates. For a real value function \(x, y, z)\) in \(\mathbb{R}^ 3\), the gradient \(∇f (x, y, z)\) is a vector value function in \(\mathbb{R}^ 3\), i.e. its value at a point \((x, y, z)\) is the vector \[onumber ∇f (x, y, z) = \left (\dfrac{∂f}{ ∂x} , \dfrac{∂f}{ ∂y} , \dfrac{∂ f}{ ∂z} \right) = \dfrac{∂f}{ ∂x}\textbf{i} + \dfrac{∂f}{ ∂y}\textbf{j} + \dfrac{∂f}{ ∂z}\textbf{k} \] in \(\math{R}^ , where each of the partial derivatives is evaluated at the point \((x, y, z)\). Thus, in this way, you can think of the symbol \(∇\) as being applied to a real value function \(f\) to produce a vector \(∇f\). It turns out that divergence and curl can also be expressed in terms of the symbol \(∇\). This is done by thinking of \(∇\) as a vector in \(\mathbb{R}^ 3\), i.e. \[∇ = \dfrac{∂}{ ∂x}\textbf{i} + \dfrac{∂}{ ∂ ∂ <9>y}\textbf{j} + \dfrac{∂}{ ∂z}\textbf{k} .\label Eq4.51}\] Here, the symbols \(\dfrac{∂}{ ∂x} \dfrac{∂}{∂y} \text{ and }\dfrac{∂}{ ∂z}\) should be considered as partial derivative operators that will be applied to a real value function , say \(f (x, y, z)\), to produce partial derivatives \(\dfrac{∂f}{ ∂x ∂ ∂ ∂ ∂} For example, \(\dfrac{∂}{ ∂x}\) applied to \(f (x, y, z) \text{ produces }\dfrac{∂f}{ ∂x}\). Is that \(∇\) really a vector? Strictly speaking, no, since \(\dfrac{∂}{ ∂x ∂ ∂ ∂ ∂} but it helps to think of \(∇\) as a vector, especially with divergence and curl, as we will soon see. The application process \(\dfrac{∂}{ ∂x ∂ ∂ ∂ ∂} z)\) is usually thought of as multiplying quantities: \[onumber \left (\dfrac{∂}{ ∂x} \right) = \dfrac{∂f}{ ∂x},\, \left (\dfrac{∂}{ ∂y} \right) = \dfrac{∂f}{ ∂y} ,\, \left (\dfrac{∂}{ ∂z} \right) = \dfrac{∂f}{ ∂z}\] For this reason, \(∇\) is often referred to as del operator, as it operates in functions. For example, it is often convenient to write the divergence div f as \(∇ \cdot \textbf{f}\), since for a vector field \(\textbf{f}(x, y, z) = f_1(x, y, z)\textbf{i}+ f_2(x, y, z)\textbf{j}+ f_3(x, y, z)\textbf{k}\), the f-point product with \(∇\) (thought of as vector) makes sense : \[onumber \start{align} ∇·\textbf{f} &= (left /dfrac{∂}{ ∂x}\textbf{i} + \dfrac{∂}{ ∂y}\textbf{j} + \dfrac{∂}{ ∂z}\textbf{k} (f_1(x, y, z)\textbf{i} + + y, z)\textbf{j} + f_3(x, y, z)\textbf{k}) \\ onumber & ;= \left (\dfrac{∂}{ ∂x} \right) (f_1) + \left (\dfrac{∂}{ ∂y} \right) (f_2) + \left (\dfrac{∂}{ ∂z} \right) (f_3) \\ onumber &= \dfracnumber &{∂f_1}{ ∂x} + \dfrac{∂f_2}{ ∂y} + \dfrac{∂f_3}{ ∂z} \\ onumber &= \text{div}\textbf{f} \\ \end{align}\] We can also write curl f in terms of \(∇\) , i.e. \(∇ × \textbf{f}\), since for a vector field \(\textbf{f}(x, y, z) = P(x, y, z)\textbf{i}+Q(x, y, z)\textbf{j}+ R(x, y, z)\textbf{k}\), we have: \[onumber \start{align} ∇ × \textbf{f} &= \start{vmatrix} \textbf{i} & \textbf{j} & \text {k} \\ \dfrac{∂}{ ∂x} & \dfrac{∂}{ ∂yy} & \dfrac{∂}{ ∂z} \\ onumber P(x , y, z) & Q(x, y, z) & R(x, y, z) \\ \end{vmatrix} \\ onumber &=\left (\dfrac{∂R}{ ∂y} − \dfrac{∂Q}{ ∂z}\∂ ∂ right } − \dfrac{∂P}{ ∂z} \right)\textbf{j} + \left (\dfrac{∂Q}{ ∂x} − \dfrac{∂P}{ ∂y} \right)\text {k} \\ onumber &= \left (\dfrac{∂R}{ ∂y} − \dfrac{∂Q}{ ∂z}\right)\textbf{i} + \left (\dfrac{∂P}{ ∂z}-\dfrac{∂R}{ ∂x} \right)\textbf{j} + \left (left { {∂Q}∂ ∂{ ∂y} \right)\textbf{k} \\ onumber &=\text{curl }\textbf{f} \\ \end{align}\] For a real value function \(f,y, z)\), the gradient \(∇f (x, y, z) = \dfrac{∂f}{ ∂x}\textbf{i} + \dfrac{∂f}{ ∂y}\textbf{j} + \dfrac{∂f}{ ∂z}\textbf{k}\) is a vector field, so we can have its divergence: \[onumber \start{align} \text{div }∇f &= ∇· ∇f \\ onumber &=\left (\dfrac{∂}{ ∂x}\textbf{i} + \dfrac{∂}{ ∂y}\textbf{j} + \dfrac{∂}{ ∂z}\textbf{k} \ right) · \left (\dfrac{∂f}{ ∂x}\textbf{i} + \dfrac{∂ ∂f}{ ∂y}\textbf{j} + \dfrac{∂f}{ ∂z}\textbf{k} \right) \\ onumber &=\dfrac{∂}{ ∂x ∂ ∂ ∂ ∂ ∂} ∂y} \right) + \dfrac{∂ <9>}{ ∂z} \left (\dfrac{∂f}{ ∂z} \right) \\ onumber &= \dfrac{∂^2 f}{ ∂x^ 2} + \\dfrac{∂^ 2 f}{ ∂y^ 2} + \dfrac{∂^ 2 f}{ ∂z^ 2} \\end{align}\] Note that this is a real value function , to which we will give a special name: Definition 4.7: Laplacian For a real value function \(f(x, y, z)\), the Laplacian of \(f\), denoted by \(Δf\), is given by \[Δf (x, y, z) = ∇· ∇f = \dfrac{∂^ 2 f}{ ∂x^ 2} + \dfrac{∂^ 2 f}{ ∂y^ 2} + \dfrac{∂^ 2 f}∂{ <1>z^ 2} .\label\{Eq4.52}\] Often the notation \(∇^2 f\) is used for laplatian instead of \(Δf\) , using the convention \(∇^ 2 = ∇· ∇\). Example 4.17 Let \(\textbf{r}(x, y, z) = x\textbf{i}+y\textbf{j}+z\textbf{k}\) is the vector position field in \(\mathbb{R}^ 3\). Next\ \lVert \textbf{r}(x, y, z)\rVert ^2 = \textbf{r} · \textbf{r} = x^ 2 + y^ 2 + z^ 2\) is an actual value function. Find the gradient of \(\lVert \textbf{r} \rVert ^2\) the divergence of \(\textbf{r}\) the bunch of the Laplacian of \(\lVert \textbf{r} \rVert ^2\) Solution: (a) \(∇ \rVert \textbf{r}\rVert ^2 = 2x\textbf{i}+2y\textbf{j}+2z\textbf{k} = 2\textbf{r}\) (b) \(∇· textbf{r} = \dfrac{∂}{ ∂x} (x)+ \dfrac{∂}{ ∂y} (y)+ \dfrac{∂}∂{ <2>z} (z) = 1+1+1 = 3\) (c) \[onumber ∇ × \textbf{r} = \start{vmatrix} \textbf{i} & \textbf{j} & \textbf{k} \\ \dfrac{∂}∂{ <1>x} & \dfrac{∂}{ ∂yy} & \dfrac{ ∂}{ ∂z} \\ x & y & z \\ \end{vmatrix} = (0−0)\textbf{i} − (0−0)\ textbf{j} + (0−0)\textbf{k} = \textbf{0} \] (d) \(Δ \lVert \textbf{r} \rVert ^2 = \dfrac{∂^ 2}{ ∂x^ 2} (x^ 2 + y^ 2 + z^ 2)+ \dfrac{∂ ∂^y^ 2} (x^ 2 + y^ 2 + z^ 2)+ \dfrac{∂^ 2}{ ∂z^ 2} (x^ 2 + y^ 2 + z^ 2) = 2+2+2 = 6\) Note that we could have calculated \(Δ\lVert \textbf{r\rVert ^2\} , using the notation \(∇\) along with parts (a) and (b): \[onumber Δ\lVert \textbf{r} \rVert ^2 = ∇· ∇ \lVert \textbf{r} \rVert ^2 = ∇· 2\textbf{r} = 2∇· \textbf{r} = 2(3) = 6\] Note that in example 4.17 if we'll get \[onumber ∇ × (∇ \rVert \textbf{rVert ^2) = 2\textbf= \rVert \textbf{r}\rVert ^2) = ∇ × 2\textbf{r} = 2∇ × \textbf{r} = 2\textbf{0} = \textbf{0} .\] The following theorem shows that this will be the case in general: Theorem 4.15. For any soft real value function \(x, y, z), ∇ × (∇f) = \textbf{0}\). Proof We see by f softly that \[\começar{align} ∇ × (∇f) &= \start{vmatrix} \textbf{i} & \textbf{j} & \textbf{k} \\ \dfrac{∂}{ ∂x} & \dfrac{∂}}{ ∂y} & \dfrac{∂}{ ∂z} \\ \dfrac{∂f}{ ∂x} & \dfrac{∂f}{ ∂y} & \dfrac{∂f}{ ∂z} \\ \end{v \\ &= \left (\dfrac{∂^ 2 f}{ ∂y∂z} − \dfrac{∂^ 2 f}{ ∂z∂y} \right)\textbf{i} − \left (\dfrac{∂^ 2 f}{ ∂x∂z } − \dfrac{∂^ 2 f}{ ∂z∂x} \right)\textbf{j} + \left (\dfrac{ ∂^ 2 f}{ ∂x∂y} − \dfrac{ ∂^ 2 f}{ ∂y∂x} \right)\textbf{k} = \textbf{0} ,\\ \end{align}\] since the mixed partial derivatives in each component are the same. \(\square\) Corollary 4.16 If a vector field \(f(x, y, z)\) has potential, then wrap \(\textbf{f} = \textbf{0}\). Another way to affirm Theorem 4.15 is that gradients are irrotationais. Also, note that in example 4.17 if we take the divergence of the r bunch we trivially get \[∇· (∇ × \textbf{r}) = ∇· \textbf{0} = 0 .\] The following theorem shows that this will be the case in general: Theorem 4.17. For any soft vector field \(\textbf{f}(x, y, z), ∇· (∇ × \textbf{f}) = 0.\) The proof is simple and left as an exercise for the reader. Corollary 4.18 The flow of the bunch of a smooth vector field \(f(x, y, z)\) across any closed surface is zero. Proof: Let \(Σ\) be a closed surface that limits a solid \(S\) The flow of \(∇ × \textbf{f}\) through \(Σ\) is \[\start{align} \iint\limits_Σ (∇ × \textbf{f})· dσ &= ∇· (∇ × \textbf{f})\, dV \text{ (by Divergence Theorem)} \\ \iiint\limits_S 0\, dV \text{ (by Theorem 4.17)} \\ &= 0 \\ \end{align}\] \(\tag{\(\textbf{QED}\)}}}} There is another method to prove theorem 4.15 that can be useful, and is often used in physics. That is, if the superficial integral \(\iint\limits_Σ f (x, y, z)\,dσ = 0\) for all surfaces \(Σ\) in some solid region (usually all of them \(\mathbb{R}^ 3\) , then we should have \(x, y, z) = 0\) in this entire region. The evidence is not trivial, and physicists don't usually bother to prove it. But the result is true, and can also be applied to double and triple integrality. For example, to prove Theorem 4.15, take that \(x, y, z)\) is a soft real value function in \(\mathbb{R}^ 3\). That \(C\) is a simple closed curve in \(\mathbb{R}^ 3\) and let \(Σ\) be any cover surface for \(C\) (i.e., \(Σ\) is guiding and its limit is \(C\)). Since \(∇f\) is a vector field, then \[onumber \begin{align} \iint\limits_Σ (∇ × (∇f)· \textbf{n}\,dσ &= \oint_C ∇f · d\textbf{r} \text{ by Stokes Theorem, so} \\ onumber &= 0 \text{ by Corollary 4.13.} \\ \end{align}\] Since the choice of \(Σ\) was arbitrary, then we should have \(∇×(∇f)·\textbf{n} = 0\) along \(\mathbb{R}^ 3\) , where n is any unit vector. Using i, j, and k instead of n, we see that we should have \(∇ × (∇f) = \textbf{0}\) in \(\mathbb{R}^ 3\), which completes the proof. Example 4.18 An electrical charge system has a load density \(ρ(x, y, z)\) and produces an electrostatic field \(\textbf{E}(x, y, z)\) at the points \(x, y, z)\) in space. The Gauss Act states that \[onumber \iint\limits_Σ \textbf{E}· dσ = 4π \iiint\limits_S ρ \,dV\] for any closed surface \(Σ\) that gladly loads, being \(S\) the solid region included by \(Σ\). Show that \(∇· \textbf{E} = 4πρ\). This is one of Maxwell's equations. Solution By Theorem of Divergence, we have \[onumber \start{align} \iiint\limits_S ∇· \textbf{E} dV &= \iint\limits_Σ \textbf{E}· dσ \\ onumber &= 4π \iiint\limits_S ρ \,dV \text{ by Gauss' Law, thus combining the integrations\ \\ onumber \iiint\limits_S (∇· \textbf{E}−4πρ) \,dV &= 0 \text{ , so} \\ the ∇number· \textbf{E}−4πρ &= 0 \text{ since }Σ \text{ and hence } S \text{ was arbitrary , so} \\ onumber ∇· \textbf{E} &= 4πρ. \\\end{align}\] Often (especially in physics) it is convenient to use other coordinate systems when dealing with amounts such as gradient, divergence, winding, and laplacian. We will present the formulas for these in cylindrical and spherical coordinates. Remember Section 1.7 that a dot ((x, y, z)\) can be represented in cylindrical coordinates \(r,φ, z), \text{ where }x = r \cos φ , y = r \sin φ, z = z.\) At each point \(r,φ, z), \text{ let }\textbf{e}_r, \textbf{e}_φ, \textbf{e}_z in the direction of increasing }r, φ, z,\) respectively (see Figure 4.6.1). Next\ \(\textbf{e}_r, \textbf{e}_φ, \textbf{e}_z\) form an orthonormal set of vectors. Note, by the right rule, that \(\textbf{e}_z × \textbf{e}_r = \textbf{e}_φ.\) Figure 4.6.1 Orthonormal vectors \(\textbf{e}_r,\, \textbf{e}_φ,\, \textbf{and}_z\) in cylindrical coordinates (left) and spherical coordinates (right). Similarly, a point \((x, y, z)\) can be represented in spherical coordinates \(ρ,φ,φ)\), where \(x = ρ \sin φ \cos φ, y = ρ \sin φ \sin φ, z = ρ \cos φ.\) At each point \(ρ,φ,φ)\), leave \(\textbf{e}_ρ, \textbf{e}_φ, \textbf{e}_φ\) be unit vectors in the direction of increasing \(ρ, φ, φ\), respectively (see Figure 4.6.2). The vectors \(\textbf{e}_ρ, \textbf{e}_φ, \textbf{and}_φ\) are orthonormal. From the rule on the right side, we see that \(\textbf{e}_φ × \textbf{e}_ρ = \textbf{e}_φ\). We can now summarize the expressions for gradient, divergence, winding and laplacian in Cartesian, cylindrical and spherical coordinates in the following tables: Cartesian \((x, y, z)\): Scalar function \(F\); Vector field \(\textbf{f} = f_1 \textbf{i}+ f_2 \textbf{j}+ f_3\textbf{k}\) gradient: \(∇F = \dfrac{∂F}{ ∂x}\textbf {i} + \dfrac{∂F}{ ∂y}\textbf{j} + \dfrac{∂F}{ ∂z}\textbf{k}\) divergence: \(∇·\textbf{f} = \dfrac {∂f_1}{ ∂x} + \dfrac{∂f_2}{ ∂y} + \dfrac{∂f_3}{ ∂z}\) curl : \(∇ × \textbf{f} = \left (\dfrac{∂f_3}{ ∂y} − dfrac{∂f_2}{ ∂z} \right) \textbf{i} + \left (\dfrac{∂f_1}{ ∂z} − \dfrac{∂f_3}{ ∂x} \right) \textbf{j} + \left (\ dfrac{∂f_2}{ ∂x} − \dfrac{∂f_1}{ ∂y} \right)\textbf{k}\) Laplacian : \(ΔF = \dfrac{∂^ 2F}{}{ ∂x^ 2} + \dfrac{∂^ 2F}{ ∂y^ 2} + \dfrac{∂^ 2F}{ ∂z^ 2}\) Cylindrical \((r,φ, φ, z)\): Scalar function \(F\); Vector field \(\textbf{f} = f_r \textbf{e}_r + f_φ \textbf{e}_φ + f_z \textbf{e}_z\) gradient: \(∇F = \dfrac{∂F}{ ∂r} \textbf{e}_r + \dfrac{1}{ r}\dfrac{ ∂F}{ ∂φ} \textbf{e }_φ + \dfrac{∂F}{ ∂z} \textbf{e}_z\) divergence: \(∇·\textbf{f} = \dfrac{1}{ r}\dfrac{ ∂}{ ∂r} (r f_r) + \dfrac{1}{ r} \dfrac{∂f_φ}{ ∂φ} + \dfrac{∂f_z}{∂z}\) curl : \(∇ × \textbf{f} = \left (\dfrac{1}{ r} \dfrac{∂f_z}{ ∂φ} − \dfrac{∂f_φ}{ ∂z} \right) \textbf{e}_r + \left (\dfrac{∂f_r}{ ∂z} − \dfrac{∂f_z}{ ∂r} \right) \textbf{e}φ + \dfrac{1} {left \ \left (dfrac{∂}∂{ (r f_ <1>φ)− \dfrac{∂f_r}{ ∂φ}\right) \textbf{and}_z\) Laplacian : \(ΔF = \dfrac{1}{ r}\dfrac{ ∂}{ ∂r} \left (r \dfrac{∂F}{ ∂r} \right) + \dfrac{1}{ r^2} \dfrac{∂^ 2F}∂φ{ ∂^ 2F}{ ∂z^ 2}\)\ \spherical ((ρ,φ,φ)\): Scalar function \(F\); Vector field \(\textbf{f} = f_ρ \textbf{e}_ρ + f_φ \textbf{e}_φ + f_φ \textbf{e}_φ\) gradient : \(∇F = \dfrac{∂F}{ ∂ρ} \textbf{e}_ρ + \dfrac{1} \dfrac{∂F}{ ∂φ} \textbf{e}_φ + \dfrac{1}{ ρ}\dfrac{ ∂F}{ ∂φ} \textbf{e}_φ\) divergence: \(∇·· \textbf{f} = \dfrac{1}{ ρ^ 2} \dfrac{∂}{ ∂ρ} (ρ^ 2 f_ρ) + \dfrac{1}{ ρ} \sin φ \dfrac{∂f_φ}{ ∂φ} + \d{1}frac{1}{ ρ \sin φ} \dfrac{∂}{ ∂φ} (\sin φ f_φ)\) curl : \(∇ × \textbf{f} { ρ \p \sin φ} \left (\dfrac{∂}{ ∂φ}{ ∂φ} (\sin φ f_φ)− \dfrac{∂f_φ}{ ∂φ} \right) \textbf{e}{ρ1{ {ρ} \ left (\dfrac{∂}{ ∂ρ} (ρ f_φ)− \dfrac{∂f_ρ}{ ∂φ} \right) \textbf{e}_φ + left (\dfrac{1}{ ρ \sin φ} \dfrac {∂f_ρ}{ ∂φ} − \dfrac{1}{ ρ} \dfrac{∂}{ ∂ρ} (ρ f_φ) \right) \textbf{e}_φ\) Laplacian : (ΔF = \dfrac{1}{ ρ^ 2} \dfrac{∂}{ ∂ρ} \left (ρ^ 2 \dfrac{∂F}{ ∂ ρ} \right) + \dfrac{1}{ ρ^ 2 \sin^2 φ} \dfrac{∂^ 2F}{ ∂φ^2} + \dfrac{1}{ ρ^ 2 \sin φ} \dfrac{∂}{ ∂φ} \left (\sin φ \ dfrac{∂F}{ ∂φ}\right) \) The derivation of the above formulas for cylindrical and spherical coordinates is simple , but extremely tedious. The basic idea is to take the Cartesian equivalent of the quantity in question and replace this formula using the appropriate coordinated transformation. As an example, we will derive the formula for the gradient in spherical coordinates. Objective: To show that the gradient of a real value function \(P,φ,φ)\) in spherical coordinates is: \[onumber ∇F = \dfrac{∂F}{ ∂ρ} \textbf{e}_ρ + \dfrac{1}{ ρ \sin φ} \dfrac{∂F}{ ∂φ} \textbf{e}_φ +\dfrac{1}{ ρ}\dfrac{ ∂F}{ ∂φ}\textbf{e}_φ\] Idea: In the Cartesian gradient formula \(∇F(x, y, z) = \dfrac{∂F}{ ∂x}\textbf{i}+ \dfrac{∂F}{ ∂y}\textbf{j}+ \dfrac{∂F}{ ∂z}\textbf{k}\) , place cartesian base vectors i, j, k in terms of spherical coordinate base vectors \(\textbf{e}_ρ, \textbf{e}_φ, \textbf{e}_φ\) and functions of \(ρ, φ \text{ and }φ\). Then place the partial derivatives \(\dfrac{∂F}{ ∂x}, \dfrac{∂F}{ ∂y} , \dfrac{∂F}{ ∂z}\) in terms of \(\dfrac{∂F}{ ∂ρ}, \dfrac{∂F}{ ∂φ} , \dfrac{∂F}{ ∂φ}\) and functions of \(ρ, φ \text{ and }φ\). Step 1: Get formulas for \(\textbf{e}_ρ, \textbf{e}_φ, \textbf{and}_φ\) in terms of i, j, k. We can see from Figure 4.6.2 that the unit vector \(\textbf{e}_ρ\) in the direction \(ρ\) at a general point \((ρ,φ,φ)\) is \(\textbf{e}_ρ = \dfrac{\textbf{r }}{ \lVert \textbf{r} \rVert } , \text{ where }\textbf{r} = x\textbf{i} + y\textbf{j} + z\textbf{k}\) is the point position vector in Cartesian coordinates. Thus, \[onumber \textbf{e}_ρ = \dfrac{\textbf{r}}{ \lVert \textbf{r} \rVert} = \dfrac{x\textbf{i}+ y\textbf{j}+ z\textbf{k}} \sqrt{ x^ 2 + y^ 2 + z^ 2}φ} y = ρ \sin \φ \sin φ, z = ρ \cos φ, \text{ and }ρ = \sqrt{ x^ 2 + y^ 2 + z^ 2}\), we have: \[onumber \fbox{\(\textbf{e}_ρ = \sin φ \cos φ\textbf{i} + \sin φ φ \textbf{j} + \cos φ\textbf{k}\)}\] Now, since the angle \(φ\) is measured in the \(x y\)-plane, then the unit vector \(\textbf{e}_φ \text{ in the direction }φ\) must be parallel to the \(x y\)-plane. That is, \(\textbf{e}_φ \text{ is from form }a\textbf{i} + b\textbf{j} + 0\textbf{k}\). To find out what \(a \text{ and }b\) are, note that since \(\textbf{e}_φ \perp \textbf{e}_ρ\), in particular \(\textbf{e}_φ \perp \textbf{e}_ρ\) when \(\textbf{e}_ρ \text{ is in }x y\)-plane. This occurs when the angle \(φ \text{ is }π/2\). Putting \(φ = π/2\) in the formula for \(\textbf{e}_ρ \text{ gives }\textbf{e}_ρ = \cos φ \textbf{i}+\sin φ \textbf {j }+0\textbf{k}\), and we see that a vector perpendicular to this is \(−\sin φ \textbf{i}+\cos φ \textbf{j}+0\textbf{k}\). Since this unit vector is also a unit vector and points in the (positive) direction \(φ\) it must be \(\textbf{e}_φ\): \[the number \fbox{\(\textbf{e}_φ = −\sin φ \textbf{i} + \cos φ\textbf{j} + 0\textbf{k}\}}\] since \(\textbf{e}_φ = = \textbf{e}_φ × \textbf{e}_ρ,\) we have: \[onumber \fbox{\(\textbf{e}_φ = \cos φ \cos φ \textbf{i} + \cos φ \sin φ \textbf{j} − \sin φ\textbf{k}\}\\\j, k in terms of \(\textbf{e}_, ρ \textbf{e}_φ, \textbf{and}_φ\). This boils down to solving a system of three equations in three unknowns. There are many ways to do this, but let's do it by combining the formulas for \(\textbf{e}_ρ \text{ and }\textbf{e}_φ \text{ to eliminate }k\), which will give us an equation involving only i and j. This, with the formula for \(\textbf{e}_φ\), will then leave us with a system of two equations in two unknowns (i and j), which we will use to solve first to j then to i. Finally, we'll settle for k. First, note that \[onumber \sin φ \textbf{e}_ρ + \cos φ \textbf{e}_φ = \cos θ \textbf{i} + \sin θ \textbf{j}\] so that \[onumber \sin θ (\sin φ \textbf{e}_ρ + \cos φ \textbf{e}_φ) + \cos \θbf{e θ}__= (\sin^2 θ +\cos^2 θ)\textbf{j} = \textbf{j} ,\] and so: \[onumber \fbox{\(\text{j} = \sin φ \sin θ \textbf{e}_ρ + \cos θ \textbf{e}_θ + \cos φ \sin θ \textbf{e}_φ\))}\Likewise, we see that \[onumber \cos θ (\sin φ \textbf{e}_ρ + \cos φ \textbf{e}_φ) − \sin θ \textbf{e}_θ = (\cos^2 θ +\sin^2 θ)\textbf{i} = \textbf{i } ,\] and so: \[onumber \fbox{\(\textbf{i} = \sin φ \cos θ \textbf{e}_ρ − \sin θ \textbf{e}_θ + \cos φ \cos θ \textbf{e}_φ\)}\] Lastly , we see that: \[onumber \fbox{\(\textbf{k} = \cos φ \textbf{e}_ρ − \sin φ \textbf{e}_φ\)}\] Step 3: Get formulas for \(\dfrac{∂F}) { ∂ρ} ∂F}{ ∂φ} , \dfrac{∂F}{ ∂φ} \text{ in terms of }\dfrac{∂F}{ ∂x} , \dfrac{∂F}{ ∂yy} , \dfrac{∂F}{ ∂z}\). By Chain Rule, we have \[onumber \start{align} \dfrac{∂F}{ ∂ρ} &= \dfrac{∂F}{ ∂x} ∂ρ} + \dfrac{∂F}{ ∂y}\dfrac{ ∂y}{ ∂ρ} + \dfrac{∂F}{ ∂z} \dfrac{∂z}{ ∂ρ}, \\ onumber \dfrac{∂F}{ ∂φ &= dfrac{∂F}{ ∂x} \dfrac{∂x}{ ∂φ} + \dfrac{∂F}{ ∂y} \dfrac{∂y}{ ∂φ} + \dfrac{∂F}{ ∂z} \dfrac{∂z}∂{ \\ onumber \dfrac{∂F}{ ∂φ} &= \dfrac{∂F}{ ∂x} \dfrac{∂x}{ ∂φ} + \dfrac{∂F}{ ∂F}{ ∂y} \dfrac{∂y}{ ∂φ} + \dfrac{∂F}{ ∂z} \dfrac{∂z}{ ∂φ} : \[\fbox{\(onumber \start{align} \dfrac{∂F}{ ∂ρ} &= \sin φ \cos φ\dfrac{'F}{ ∂F}{ ∂y} + \cos φ \dfrac{∂F}{ ∂z} \\ onumber \dfrac{∂F}{ ∂φ} &= −ρ \sin φ \sin φ \dfrac{∂F}{ ∂x} + ρ \sin φ \cos φ \dfrac{∂F}{ ∂y} \\ onumber \dfrac{∂F}{ ∂φ} &?ρ \cos φ \cos φ \dfrac{∂F}{ ∂x} v ρ \cos φ \sin φ \dfrac{∂F}{ ∂y} − ρ \sin φ \dfrac{∂F}{ ∂z} \\ \end{align}\\\\] Step 4 : Use the three step 3 formulas to resolve to \(\dfrac{∂F}{ ∂x} , \dfrac{∂F}{ ∂y ∂ ∂}}\) in terms of \(\dfrac{∂F}{ ∂ρ} , \dfrac{∂F}{ ∂φ} , \dfrac{∂F}{ ∂φ} \). Again, this involves solving a system of three equations in three unknowns. Using a similar elimination process as in Step 2, we have: \[\fbox{\(onumber \start{align}\dfrac{∂F}{ ∂x} &= \dfrac{1}{ ρ \sin φ} \left (ρ \sin^2 φ \cos φ \dfrac{∂F}{ ∂ρ} − \sin φ \dfrac{∂F}{ ∂φ} + \sin φ \cos φ \cos φ \dfrac {∂F}{ ∂φ}\right) \\ onumber \dfrac{∂F}{ ∂y} &= \dfrac{1}{ ρ \sin φ} \left (ρ \sin^2 φ \sin φ \dfrac{∂F}{ ∂ρ} + \cos φ \dfrac{∂F}{ ∂φ} + \sin φ \cos φ \sin φ \ dfrac{∂F}{ ∂φ}\right) \\ onumber \dfrac{∂F}{ ∂z} &= \dfrac{1}{ρ} \left (ρ \cos φ \dfrac{∂F}{ ∂ρ} − \sin φ \dfrac{∂F}{ ∂φ}\right) \\ \end{align}\}}}\] Step 5: Replace formulas with i, j, step 2 k, and the formulas for Step 4 \(\dfrac{∂F}{ ∂x} , \dfrac{∂F}{ ∂y} , step 4 \dfrac{∂F}{∂z}\) for the Cartesian gradient formula \(∇F(x , y, z) = \dfrac{∂F}{ ∂x} \textbf{i}+ \dfrac{∂F}{ ∂y} \textbf{j}+ \dfrac{∂F}{ ∂z} \textbf{k}\). Making this last step is perhaps the most tedious, because it involves simplifying the terms \(3×3+3×3+ 2×2 = 22\) That is, \[onumber \start{align} ∇F &= \dfrac{1}{ ρ \sin φ} \left (ρ \sin^2 φ \cos φ \dfrac{∂F}{ ∂ρ} −−sin φ \dfrac{∂F}{ ∂φ} \\sin φ cos φ \cos φ \dfrac{∂F}{ ∂φ}\right) (\sin φ \cos φ \textbf{e}_ρ −\sin φ \textbf{e}_φ +\cos φ \cos φ \textbf{e}_φ) \\ onumber &+\dfrac{1}{ ρ \sin φ} \left (ρ \sin^2 φ \sin φ \dfrac{∂F}{ ∂ρ} +\cos φ \dfrac{∂F}{ ∂φ}

+\sin φ \cos φ \sin φ \dfrac{∂F}{ ∂φ}\right) (\sin φ \sin φ \textbf{e}_ρ +\cos φ \textbf{e}_φ +\cos φ \sin φ \textbf{e}_φ)\\ onumber &+\dfrac{1}{ ρ} \left (ρ \cos φ \dfrac{∂F}{ ∂ρ} −\sin φ \dfrac{∂F}{ ∂φ}\right) (\cos φ \textbf{e}_ρ{e}_ρ_e}_ρ_ φ \textbf{e}_φ) ,\\ \end{align}\] we see has 8 terms involving \(\textbf{e}_ρ\), 6 terms involving \(\textbf{e}_φ\) and 8 terms involving \(\textbf{e}_φ\). But algebra is simple and produces the desired result: \[∇F = \dfrac{∂F}{ ∂ρ} \textbf{e}_ρ + \dfrac{1}{ ρ \sin φ} \dfrac{∂F}{ ∂φ} \textbf{e}_φ + \dfrac{1}{ ∂F}{ ∂φ} \textbf{e}_φ \quad \checkmark \] Example 4.1 9 In example 4.17 we show that \(∇\lVert \textbf{r} \rVert ^2 = 2\textbf{r} \text{ and }Δ\lVert \textbf{r\rVert ^2 = 6= 6= 6= 6= 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6= 6= 6= 6= 6= 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = 6 = , \text{ where }\textbf{r}(x, y, z) = x\textbf{i} + y\textbf{j} + z\textbf{k}\) in Cartesian coordinates. Make sure we have the same answers if we switch to spherical coordinates. Solution Since \(\lVert \textbf{r} \rVert ^2 = x^ 2 + y^ 2 + z^ 2 = ρ^ 2 \text{ in spherical coordinates, let }F(ρ,φ,φ) = ρ^ 2 \) (so that \(f(ρ,φ,φ) = \lVert \textbf{r} \rVert ^2\)). The gradient of \(F\) in spherical coordinates is \[onumber \start{align} ∇F &= \dfrac{∂F}{ ∂ρ} \textbf{e}_ρ + \dfrac{1}{ ρ \sin φ} \dfrac{∂F}{ ∂φ} \text{e}_φ + \dfrac{1}{ ρ} \dfrac{∂F}{ ∂φ \} textbf{e}_φ onumber &=2ρ \textbf{e}_ρ + \dfrac{1}{ ρ \sin φ} (0)\textbf{e}_φ + \dfrac{1}{ 2ρ} (0)\textbf{e}_φ \\ onumber &= 2ρ \textbf{e}_ρ = 2ρ \dfrac{\textbf{r}}{ \lVert \textbf{r} \rVert} , \text{ as we showed earlier, so}\\ onumber &= 2ρ \dfrac{\textbf{r}}{ ρ} = 2\textbf{r} , \text{ as expected. And laplacian is} \\ onumber ΔF &= \dfrac{1}{ ρ^ 2} \dfrac{∂}{ ∂ρ} \left (ρ^2 \dfrac{∂F}{ ∂ρ} \right) + \dfrac{1}{ ρ^ 2 \sin^2 φ} \dfrac{∂^2F}{ ∂φ^2} + \dfrac{1}{ ρ^ 2 \sin φ} \dfrac{∂ }{ ∂φ ∂}{ <6>} \left (\sin φ \dfrac{∂F}{ ∂φ}\right) \\ onumber &= \dfrac{1}{ ρ^ 2} \dfrac{∂}{ ∂ρ} (ρ^ 2 2ρ) + \dfrac{1}{ ρ^ 2 \sin φ} (0) + \dfrac{1}{ ρ^ 2 \sin φ \} dfrac{∂}{ ∂φ} \left (\sin φ(0)\right) \\ onumber &= \dfrac{1}{ ρ^ 2} \dfrac{∂}{ ∂ρ} (2ρ^ 3) + 0 + 0 \\ onumber &= \dfrac{1}{ ρ^ 2} (6ρ^ 2) = 6 , \text{ as expected.} \\ \\end\\[